Section 2.1 – Relations and Functions

- We use graphs in Math, like History books use pictures. Graphs give us a quick way to make comparisons, draw conclusions, and approximate quantities. The next section will involve different names of graphing relationships and how to plot and read information.

Coordinate System

- Similar to a real number on a real number line, ordered pairs can be represented by points on the Cartesian Plane.
- Ordered pairs are written in the form \((x, y)\)
- There is a unique point on the plane that corresponds to every ordered pair

- The ordered pair \(A (0, 0)\) is located at the origin.
- The ordered pair \(B (3, 2)\) is located three units to the right and two units up from the origin.
- The ordered pair \(C (-3, 0)\) is located three units to the left of the origin on the \(x - axis\).
- The ordered pair \(D (0, 3)\) is located three units up from the origin on the \(y - axis\).
- The ordered pair \(E (-3, -4)\) is located three units to the left and four units down from the origin.
- The ordered pair \(F (4, -3)\) is located four units to the right and three units down from the origin.
- The ordered pair \(G (-3, 4)\) is located three units to the left and four units up from the origin.

Ordered pairs \((4, -3)\) and \((-3, 4)\) plot different points. That is why they are called ordered pairs, it makes a significant difference which number comes first.
Relations

- Relations are sets of **ordered pairs** \((x, y)\)
- The set of the first components, or \(x\) — **values**, is the **DOMAIN**
- The set of the second components or \(y\) — **values**, is called the **RANGE**
- To find solutions to a relation, values are arbitrarily assigned for the \(x\) **term** from the set of real numbers.
- This makes \(x\) the **independent variable**
- Choosing input values for \(x\) provides us with output values for \(y\)
- This makes \(y\) the **dependant variable**

**Example:**

<table>
<thead>
<tr>
<th>Input (x)</th>
<th>Relation (y = 2x + 1)</th>
<th>Output (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>(2(-3) + 1)</td>
<td>-5</td>
</tr>
<tr>
<td>0</td>
<td>(2(0) + 1)</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(2(2) + 1)</td>
<td>5</td>
</tr>
</tbody>
</table>

These values represent **3 solutions** to the **infinitely many** for the relation \(y = 2x + 1\).

These solutions can be represented as:

1. Ordered pairs: \((-3, -5), (0, 1), (2, 5)\)
2. In a table:

\[
\begin{array}{ccc}
  x & -3 & 0 & 2 \\
  y & -5 & 1 & 5 \\
\end{array}
\]

3. **Using Mapping Notation**

4. Or by graphing
Example 1: Determine the Domain and the Range of the following ordered pairs

\[ A = \{(1, 2), (-3, 5), (4, -2)\} \]
\[ B = \{(-3, 4), (1, 0), (0, 2), (3, 2)\} \]
\[ C = \{(-2, 1), (1, 0), (3, 3), (1, 4)\} \]
\[ D = \{(-3, -1), (-3, 3), (2, 3), (4, 0)\} \]

Solution 1:

Domain of \( A \) = \{-3, 1, 4\} \hspace{1cm} Range of \( A \) = \{-2, 2, 5\}
Domain of \( B \) = \{-3, 0, 1, 3\} \hspace{1cm} Range of \( B \) = \{2, 0, 4\}
Domain of \( C \) = \{-2, 1, 3\} \hspace{1cm} Range of \( C \) = \{0, 1, 3, 4\}
Domain of \( D \) = \{-3, 2, 4\} \hspace{1cm} Range of \( D \) = \{-1, 0, 3\}

Example 2: Determine the Domain and the Range of the graphs below

a) ![Graph](image1.png) \hspace{1cm} b) ![Graph](image2.png) \hspace{1cm} c) ![Graph](image3.png)

Solution 2:

<table>
<thead>
<tr>
<th></th>
<th>Domain:</th>
<th>Range:</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>All Real Numbers</td>
<td>( y \geq -2)</td>
</tr>
<tr>
<td>b</td>
<td>(-2 \leq x \leq 2)</td>
<td>(0 \leq y &lt; 4)</td>
</tr>
<tr>
<td>c</td>
<td>(x \geq -2)</td>
<td>All Real Numbers</td>
</tr>
<tr>
<td>d</td>
<td>{0, 1, 2, 3}</td>
<td>{0, 1, 2, 3}</td>
</tr>
<tr>
<td>e</td>
<td>(-2 \leq x \leq 2)</td>
<td>(-4 \leq y \leq 4)</td>
</tr>
<tr>
<td>f</td>
<td>All Real Numbers</td>
<td>All Real Numbers</td>
</tr>
</tbody>
</table>
Functions

- A function is a special type of relation

Function

- For every value of the domain \(x - value\), there is one and only one, value for the range \(y - value\)
- Each element in the domain corresponds to exactly one element in the range.

One-to-One Function

- A function in which every individual value of the domain \(x - value\) is associated with one value of the range \(y - value\), and vice versa.
- This means that if the function is a one-to-one function, then for each \(x\) in the domain, there is one, and only one, \(y\) in the range, and no \(y\) in the range is the image of more than one \(x\) in the domain.

Hierarchy of Relations, Functions, and One-to-one Functions

<table>
<thead>
<tr>
<th>Relation</th>
<th>Function</th>
<th>One-to-one Function</th>
</tr>
</thead>
</table>

Example:

- 1-1: A function, not 1-1: Not a function, just a relation

Note: The Range (Output) depends on the Domain (Input)
Example: Given the ordered pairs: \((-5, 4), (-3, 2), (-2, 0), (0, -2), (1, -3), (4, -4)\), what is the value of \(y\) (output) when \(x\) (input) is 0?

Solution: From \((0, -2)\) the output is \(-2\) or \(y = -2\)

**Vertical Line Test for Functions**

- An equation defines \(y\) as a function of \(x\) if and only if every **vertical line** in the coordinate plane intersects the **graph** of the equation **only once**.

**Horizontal Line Test for One-to-One Functions**

- A function \(y\) is a one-to-one function of \(x\) if and only if every **horizontal line** in the coordinate plane intersects the **function** at most **only once**.

Example: State whether the following relations is a function, a one-to-one function, or neither.

![Graphs](image)

a) A **vertical line** intersects the graph **once** so it is a **function**. A **horizontal line** intersects the graph **once**, therefore it is a **one-to-one function**.

b) A **vertical line** intersects the graph **once**, so it is a **function**. A **horizontal line** intersects the graph **more than once**, therefore the graph is **not a one-to-one function**.

c) A **vertical line** intersects the graph **more than once**, so it is **not a function**, just a **relation**.
Section 2.1 – Practice Questions

Without plotting on a grid, which quadrant do the following points belong to?

1. (4, -2) \[ \checkmark \]  
2. (6, 3) \[ \checkmark \]  
3. (-1, 3) \[ \checkmark \]  
4. (-2, -6) \[ \checkmark \]  
5. (-3, 0) [no quadrant] \[ \checkmark \]  
6. (0, 0) [no quadrant] \[ \checkmark \]

7. Plot the points of the grid provided

<table>
<thead>
<tr>
<th>A(-3, 1)</th>
<th>B(-4, -2)</th>
<th>C(-5, 0)</th>
<th>D(0, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(3, -5)</td>
<td>F(4, 3)</td>
<td>G(4, 0)</td>
<td>H(0, -4)</td>
</tr>
</tbody>
</table>

8. A relation is:
   a) Any set of ordered pairs
   b) Two sets of ordered pairs that are related
   c) A graph of ordered pairs
   d) A set of ordered pairs where the domain corresponds to exactly one range

9. A function is:
   a) Any set of ordered pairs
   b) A set or ordered pairs in which a value in the domain corresponds to exactly one value in the range
   c) A set of ordered pairs in which a value in the range corresponds to exactly one value in the domain
   d) A graph of ordered pairs
Use the vertical line test to determine if the following are relations or functions

10. [Graph with a parabola, marked as a function]
11. [Graph with a hyperbola, marked as a function]
12. [Closed circle graph, marked as not a function]
13. [Graph with a curve, marked as a function]
14. [Graph with a curve, marked as a function]
15. [Graph with a curve, marked as a function]

Do the mapping notations into functions, 1-1 functions, or neither?

16. [Diagram with arrows, marked as a 1-1 function]
17. [Diagram with arrows, marked as a function]
18. [Diagram with arrows, marked as not a function]
19. [Diagram with arrows, marked as not a function]
Section 2.2 – Linear and Non-Linear Equations

Expressions

- An expression is a collection of numbers, variables and operation signs
- Expressions DO NOT have an EQUALS SIGN

Example:

a) 5
b) 2x - 3
c) 3x^2 + 2x - 5
d) \sqrt{5}

Equations

- An equation is a mathematical statement that two expressions are equivalent
- There is an EQUALS SIGN

Example:

a) y = 2
b) y = 3x + 4
c) x + 4y = 7
d) x = y^2

Linear Equations

- A linear equation is any equation of the form \( Ax + By = C \), where \( A, B, \) and \( C \) are constants and \( x \) and \( y \) are variables. All linear equations are functions except a vertical line.

Example:

- \( x = 3 \) \( \rightarrow \) Not a function
- \( y = -1 \) \( \rightarrow \) A function
- \( 2x + 3y = 5 \) \( \rightarrow \) A function
Graphing linear Equations of the Type $Ax + By = C$

1. To find the $y$ - intercept (where the line crosses the $y$ - axis), set $x = 0$ and solve for $y$.
   To find the $x$ - intercept (where the line crosses the $x$-axis), set $y = 0$ and solve for $x$.
2. To get a third point, pick another value for $x$, and solve for $y$.
3. Plot the three points from steps 1 and 2 and draw a straight line through the points.

**Example:** Graph $2x + 3y = 6$

**Solution:** Three points picked: Solve for three missing values:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$3x + 2y = 6$</th>
<th>$3x + 2y = 6$</th>
<th>$3x + 2y = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$3(0) + 2y = 6$</td>
<td>$3x + 2(0) = 6$</td>
<td>$3(-2) + 2y = 6$</td>
</tr>
<tr>
<td>$\text{y} = 3$</td>
<td>$x = 2$</td>
<td>$y = 6$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Therefore, the ordered pairs are:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>Plot these three points: $(0, 3), (2, 0), (-2, 6)$, and draw a straight line through the three points. <strong>Extend the line in both directions.</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>Tabular</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>Tabular</td>
</tr>
<tr>
<td>$-2$</td>
<td>6</td>
<td>Tabular</td>
</tr>
</tbody>
</table>

**Summary of Ordered Pair $(x, y)$**

- **Domain:** Input
- **Range:** Output
- **Independent Variable:** Dependent Variable
Graphing Linear Equations of the Type \( y = mx + b \)

1. Identify the \( y \)-intercept, plot that point.
2. Identify the Slope in the given equation and trace it to your next point, plot that
3. Repeat step #2
4. Connect the points to create your line.

\[ m = \text{Slope} = \frac{\text{Rise}}{\text{Run}} \quad b = y \text{-intercept} \]

OR

1. Select three values of \( x \) that are multiples of the denominator of the slope.
2. Solve for \( y \) in each case.
3. Plot three points from steps 1 and 2. Draw a straight line through the points

**Example:** Graph \( y = -\frac{2}{3}x + 4 \)  
(Slope is the constant in front of the \( x \))

**Solution:** Three points picked: Solve for three missing values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>-3</td>
<td>6</td>
</tr>
</tbody>
</table>

\[ y = -\frac{2}{3}x + 4 \quad y = -\frac{2}{3}(0) + 4 \quad y = -\frac{2}{3}(-3) + 4 \]

\[ y = -\frac{2}{3}x + 4 \quad y = -\frac{2}{3}(0) + 4 \quad y = -\frac{2}{3}(-3) + 4 \]

Therefore, the ordered pairs are:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>-3</td>
<td>6</td>
</tr>
</tbody>
</table>

Plot these three points: \((0, 4), (3, 2), (-3, 6)\), and draw a straight line through the three points. **Extend the line in both directions.**
Non-Linear Equations

Rules for graphing Non-Linear Equations

1. Use positive numbers, negative numbers, and zero whenever possible.
2. If any value is to an even power both positive and negative values must be used.
3. Use values between 0 and 1 when the variable is in the denominator, or is in the exponent

Example: Is \( x = y^2 \) a function?

Solution: Since \( y \) is an even power, positive and negative values of \( y \) are used.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>-1</th>
<th>2</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-2</td>
</tr>
</tbody>
</table>

- The graph is not a function since it does not pass the vertical line test.

Example: Is \( y = x^2 - 1 \) a function?

Solution: Since \( x \) is an even power, positive and negative values of \( x \) are used.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>-1</th>
<th>2</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

- The graph is a function since it does pass the vertical line test.
Example: Is $x^2 + y^2 = 9$ a function?

Solution: Since $x$ and $y$ is an even power, positive and negative values of $x$ and $y$ are used:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>±3</th>
<th>1</th>
<th>±$\sqrt{8}$</th>
<th>3</th>
<th>0</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>±3</td>
<td>1</td>
<td>±$\sqrt{8}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- The graph is not a function since it does not pass the vertical line test.

Example: Is $y = 2^x$ a function?

Solution: Since $x$ is in the exponent, positive and negative values of $x$ are used:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

- The graph is a function since it does pass the vertical line test.
Example: Is $y = \frac{1}{x}$ a function?

Solution: Since $x$ is in the denominator, values between 0 and 1 must be used

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\frac{1}{2}$</th>
<th>$-\frac{1}{2}$</th>
<th>$\frac{1}{4}$</th>
<th>$-\frac{1}{4}$</th>
<th>→</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\frac{1}{2}$</th>
<th>$-\frac{1}{2}$</th>
<th>$\frac{1}{4}$</th>
<th>$-\frac{1}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{2}$</td>
<td>-1</td>
<td>$\emptyset$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{3}$</td>
<td>2</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
<td>$-\frac{1}{4}$</td>
</tr>
</tbody>
</table>

- The graph is a function since it does pass the vertical line test.
- $x$ cannot be zero because $1/0$ does not exist (asymptote)
- $y$ cannot be zero because 1 divided by a very large number is a very small number, but still not zero.

Domain and Range Refresher

Domain: The Domain of the graph is the representation of every possible $x$ – value (input) that the graph contains

Range: The Range of the graph is the representation of every possible $y$ – value (output) that the graph contains
Section 2.2 – Practice Questions

1. The domain of a relation is:
   a) The set of all \( x \) and \( y \) values in ordered pairs
   b) The sum of the components in the ordered pairs
   c) The set of all the first components in the ordered pairs
   d) The set of all the second components on the ordered pairs

2. The range of a relation is:
   a) The set of all \( x \) and \( y \) values in ordered pairs
   b) The sum of the components in the ordered pairs
   c) The set of all the first components in the ordered pairs
   d) The set of all the second components on the ordered pairs

Determine whether the given ordered pair is a solution to the equation (a point on the line).

3. \((2,3); \ 3x - 5y = -9\) \(\checkmark\)

4. \((0,4); \ y = -\frac{1}{3}x + 4\) \(\checkmark\)

5. \((1,-1); \ 3y = 5 - 2x\) \(\boxtimes\)

6. \((6,8); \ \frac{1}{3}x - \frac{1}{4}y = 4\) \(\checkmark\)

7. \((4,2); \ x = 4\) \(\checkmark\)

8. \((-1,3); \ y = -1\) \(\blacksquare\)

9. \((4,-3); \ 0.05x - 1.2y = 3.8\) \(\checkmark\)

10. \((\frac{2}{3},-\frac{3}{4}); \ 60x - 36y = 13\) \(\checkmark\)
Graph the following Linear Equations

11. $2x + 3y = 6$

12. $2x + y = -4$

13. $2x - \frac{1}{2}y = 2$

14. $3x + 2y = 5$

15. $\frac{2}{3}x - 0.4y = 2$

16. $y = -2x - 1$

Rearrange

Rise →

Run →

Adrian Herlaar, School District 61
17. \[ y = -\frac{3}{4}x + 1 \]

18. \[ y = \frac{2}{3}x - 2 \]

19. \[ \frac{1}{2}x + 0.6y = 3 \]

20. \[ -0.4x + \frac{1}{3}y = 1 \]

21. \[ x = 3 \]

22. \[ y = -2 \]
Graph the following Non-Linear Equations

23. \( y = x^2 + 1 \)

\[
\begin{array}{c|ccccc}
 x & 0 & 1 & -1 & -5 & 5 \\
 y & 1 & 2 & 2 & 1 & 1 \\
\end{array}
\]

24. \( x = \frac{1}{3}y^2 \)

\[
\begin{array}{c|cccc}
 x & 0 & 3 & 3 & 12 & 12 \\
 y & 0 & 3 & -3 & 6 & -6 \\
\end{array}
\]

25. \( x = 2^y \)

\[
\begin{array}{c|cccc}
 x & 0 & 2 & -2 & 2 & -2 \\
 y & 0 & 1 & 1 & 5 & 5 \\
\end{array}
\]

26. \( y = \frac{1}{x} \)

\[
\begin{array}{c|cccc}
 x & 0 & 1 & -2 & 1 & -2 \\
 y & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \\
\end{array}
\]
27. \( y = x^2 - 2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>-1</th>
<th>-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>-2</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

28. \( x = y^2 - 2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>2</th>
<th>-1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

29. \( y = (x - 2)^2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

30. \( x = (y - 2)^2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>4</th>
<th>4</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
31. \( y = 4 - \frac{1}{2}x^2 \)

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & 0 & 2 & -4 & -2 & -4 \\
\hline
y & 4 & 2 & 4 & 2 & -4 \\
\hline
\end{array}
\]

32. \( y = \frac{1}{2}x^3 \)

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & -2 & -1 & 0 & 1 & 2 \\
\hline
y & -4 & \frac{1}{2} & 0 & \frac{1}{2} & 4 \\
\hline
\end{array}
\]

33. \( y = \sqrt{x} \)

\[
\begin{array}{|c|c|c|c|}
\hline
y & \sqrt{4} & \sqrt{9} & \sqrt{16} & \sqrt{25} \\
\hline
\end{array}
\]

34. \( (x - 1)^2 + y^2 = 16 \)

\[
\begin{array}{|c|c|c|c|}
\hline
x & 1 & -3 & 5 & ? & ? \\
\hline
y & 4 & 0 & 0 & ? & ? \\
\hline
\end{array}
\]
Section 2.3 – Arithmetic Sequence and Series

A Sequence

- A sequence is simply a list of numbers
- Each number in the list is called a Term
- They are listed: first term, second term, third term, and so on...
- Sequences can be finite (they end) or infinite (they don’t end)

Unlike graphing equations and using function notation of $x$ and $y$ values sequences use script notation.

Example:

Term 1 is known as: $a_1$ or $t_1$

Term 2 is known as: $a_2$ or $t_2$

Term 3 is known as: $a_3$ or $t_3$

... ...

The $n^{th}$ is known as: $a_n$ or $t_n$

Sequence

A finite sequence is a function for which the domain ($x$ – values) is a subset of the natural numbers: \{1, 2, 3, ..., $n$\} for some finite number $n$

An infinite sequence is a function for which the domain ($x$ – values) is the set of natural numbers: \{1, 2, 3, ...\}

Example: Write the first four terms of the sequence

\(a) \quad a_n = \frac{n+1}{n}\)
\(b) \quad b_n = 2n - 3\)
\(c) \quad t_n = 2^n\)

Solution:

\(a) \quad a_1 = \frac{1+1}{1} = 2, \quad a_2 = \frac{2+1}{2} = \frac{3}{2}, \quad a_3 = \frac{3+1}{3} = \frac{4}{3}, \quad a_4 = \frac{4+1}{4} = \frac{5}{4}\)

\(b) \quad b_1 = 2(1) - 3 = -1, \quad b_2 = (2)(2) - 3 = 1, \quad b_3 = (2)(3) - 3 = 3, \quad b_4 = (2)(4) - 3 = 5\)

\(c) \quad t_1 = 2^1 = 2, \quad t_2 = 2^2 = 4, \quad t_3 = 2^3 = 8, \quad t_4 = 2^4 = 16\)
**Arithmetic Sequence**

- When we have a **sequence in which the successive terms** have a **common difference**, the sequence is called and **arithmetic sequence**
- For example the sequence, 3, 7, 11, 15, ... has a **common difference** of 4. Every next term is achieved by **adding** 4 to the term previous.
- The common difference, \( d \), of this sequence is 4.

If we look at the pattern we may see something helpful...

\[
\begin{align*}
1^{st} \text{ term:} & \quad a_1 = a_1 \\
2^{nd} \text{ term:} & \quad a_2 = a_1 + d \\
3^{rd} \text{ term:} & \quad a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d \\
4^{th} \text{ term:} & \quad a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d
\end{align*}
\]

- From this pattern we are able to generate the general equation of an **Arithmetic Sequence**

**The \( n^{th} \) term of an Arithmetic Sequence**

- For an arithmetic sequence \( \{t_n\} \) whose first term is \( a \), with a **common difference** \( d \):

\[
t_n = a + (n - 1)d \quad \text{for any integer } n \geq 1
\]

**Example:** For each arithmetic sequence, identify the common difference.

a) 3, 5, 7, 9, ...
b) 11, 8, 5, 2, ...

**Solution:**

a) \[ 5 - 3 = 2, \quad 7 - 5 = 2, \quad 9 - 7 = 2, \quad \text{Therefore } d = 2 \]
b) \[ 8 - 11 = -3, \quad 5 - 8 = -3, \quad 2 - 5 = -3, \quad \text{Therefore } d = -3 \]

**Example:** Determine if the sequence \( \{t_n\} = \{3 - 2n\} \) is arithmetic

**Solution:**

\[
\begin{align*}
t_1 &= 3 - 2(1) = 1 \\
t_2 &= 3 - 2(2) = -1 \\
t_3 &= 3 - 2(3) = -3
\end{align*}
\]

\[
1, -1, -3, \ldots \text{ has a common difference of } -2
\]

So the sequence is arithmetic!
Example: Find the 12th term of the arithmetic sequence 2, 5, 8, ...

Solution: 
\[ a = 2 \quad d = 3 \]
\[ t_n = a + (n - 1)d \]
\[ t_{12} = 2 + (12 - 1)3 \quad \rightarrow \quad 35 \]

Example: Which term in the arithmetic sequence 4, 7, 10, ... has a value of 439?

Solution: 
\[ d = 7 - 4 = 3 \]
\[ t_n = a + (n - 1)d \]
\[ 439 = 4 + (n - 1)3 \]
\[ 435 = (n - 1)3 \quad \rightarrow \quad 145 = n - 1 \]
\[ n = 146 \quad \text{The 146th term is 439.} \]

Example: The 7th term of an arithmetic sequence is 78, and the 18th term is 45. Find the 1st term.

Solution: 
There are 18 - 7 = 11 terms between 45 and 78. And the difference between them is 45 - 78 = -33

So,
\[ 11d = -33 \quad \rightarrow \quad d = -3 \]

So,
\[ t_n = a + (n - 1)d \]
\[ t_7 = a + (7 - 1)(-3) \]
\[ 78 = a + (-18) \]
\[ 78 + 18 = a = 96 \]

Example: Find x so that \(3x + 2, 2x - 3, \text{ and } 2 - 4x\) are consecutive terms of an arithmetic sequence.

Solution: Since they are consecutive,
\[(2x - 3) - (3x + 2) = d \quad \text{and} \quad (2 - 4x) - (2x - 3) = d\]

So since they both equal \(d\), we can set them equal to each other.
\[ (2x - 3) - (3x + 2) = (2 - 4x) - (2x - 3) \]
\[ 2x - 3 - 3x - 2 = 2 - 4x - 2x + 3 \]
\[ -x - 5 = -6x + 5 \quad \rightarrow \quad 5x = 10 \quad \rightarrow \quad x = 2 \]
Arithmetic Series

- An arithmetic series is when we take our given sequence and we add it all together. We have finite and infinite sums just like we have for sequences, but we’re only going to look at finite series.
- Here’s the formula:

\[ S_n = \frac{n}{2} (a + l) \]

\[ or \]

\[ S_n = \frac{n}{2} (2a + (n - 1)d) \]

Where \( a = the \ first \ term, \ l = the \ last \ term, \) and \( d = the \ common \ difference \)

- We can interchange the two equations, depending on what information is given to us.
- Then it really just becomes plug by numbers.

**Example:** Find the sum of the positive integers from 1 to 50 inclusive.

**Solution:**

\[ a = 1, \ l = 50, \ d = 1 \]

\[ S_n = \frac{n}{2} (a + l) \]

\[ S_{50} = \frac{50}{2} (1 + 50) \rightarrow 25(51) \rightarrow 1275 \]

**Example:** Find the sum of the first 25 terms of the series 11 + 15 + 19 + …

**Solution:** The series is arithmetic (has a common difference) with \( a = 11, d = 4, and n = 25 \)

\[ S_n = \frac{n}{2} (2a + (n - 1)d) \]

\[ S_{25} = \frac{25}{2} (2(11) + (25 - 1)4) \rightarrow 12.5(22 + 96) \rightarrow 1475 \]
Example: Find the sum of the series $7 + 10 + 13 + \cdots + 100$.

Solution: $a = 7, l = 100, d = 3$ but We don't know $n$, so we solve for that first

To find $n$ we use the formula from Section 2.2

$$t_n = a + (n - 1)d$$

Since we want to know how many terms there are and 100 is the last term, if we solve for that we'll get $n$.

$$100 = 7 + (n - 1)(3)$$
$$\rightarrow 100 = 7 + 3n - 3$$
$$\rightarrow 100 = 4 + 3n$$
$$\rightarrow 96 = 3n$$
$$n = 32$$

Now we can solve for the sum since we know $n$.

$$S_n = \frac{n}{2}(a + l)$$

$$S_{32} = \frac{32}{2}(7 + 100)$$

$$S_{32} = 16(107)$$

$$S_{32} = 1712$$

Example: Find the sum of the $5 + 9 + 13 + \cdots + 137$

Solution: $a = 5, l = 137, d = 4$ but We don't know $n$, so we solve for that first

To find $n$ we use the formula from Section 2.2

$$t_n = a + (n - 1)d$$

Since we want to know how many terms there are and 137 is the last term, if we solve for that we'll get $n$.

$$137 = 5 + (n - 1)(4)$$
$$\rightarrow 137 = 5 + 4n - 4$$
$$\rightarrow 137 = 1 + 4n$$
$$\rightarrow 136 = 4n$$
$$n = 34$$

Now we can solve for the sum since we know $n$.

$$S_n = \frac{n}{2}(a + l)$$

$$S_{34} = \frac{34}{2}(5 + 137)$$

$$S_{34} = 17(142)$$

$$S_{34} = 2414$$
Section 2.3 – Practice Problems

Write the first four terms of each of the following sequences

1. \( n^2 - 2 \)
   - \( t_1, t_2, t_3, t_4 \)
   - 1, 2, 3, 4

\( 2^2 = 4 - 2 \)
\( 3^2 = 9 - 2 \)
\( 4^2 = 16 - 2 \)

2. \( \frac{n+2}{n+1} \)
   - \( 3, 4, 5, 6 \)
   - \( 2, 3, 4, 5 \)

\( \frac{-1, 2, 7, 14}{-1, 2, 7, 14} \)

3. \( \{(-1)^{n+1}n^2\} \)
   - \( 1, 2, 3, 4, n \)

\( (-1)(1) = -1 \)
\( (-1)(2) = -2 \)
\( (-1)(3) = -3 \)
\( (-1)(4) = -4 \)

4. \( \frac{3^n}{2n+1} \)
   - \( 1, 2, 3, 4, n \)

\( \frac{3^1}{2(1)+1} = \frac{3}{3} = 1 \)
\( \frac{3^2}{2(2)+1} = \frac{9}{5} \)
\( \frac{3^3}{2(3)+1} = \frac{27}{7} \)
\( \frac{3^4}{2(4)+1} = \frac{81}{9} = 9 \)

5. \( \left\{ \frac{2^n}{n^2} \right\} \)
   - \( 1, 2, 3, 4 \)

\( \frac{2^1}{1^2} = \frac{2}{1} \)
\( \frac{2^2}{2^2} = \frac{4}{4} \)
\( \frac{2^3}{3^2} = \frac{8}{9} \)
\( \frac{2^4}{4^2} = \frac{16}{16} \)

6. \( \left\{ \frac{2^n}{3} \right\} \)
   - \( 1, 2, 3, 4 \)

\( \frac{2^1}{3} = \frac{2}{3} \)
\( \frac{2^2}{3} = \frac{4}{3} \)
\( \frac{2^3}{3} = \frac{8}{3} \)
\( \frac{2^4}{3} = \frac{16}{3} \)

Find the indicated arithmetic term

7. \( a = 5, d = 3, \text{find } t_{12} \)
   - \( n = 12 \)
   - \( n = 12, \text{ find } t_{12} \)
   - \( t_{12} = S(n-1)(3) \)
   - \( t_{12} = S(12-1)(3) \)
   - \( t_{12} = 38 \)

\( t_{12} = S(11)(3) \rightarrow 33 \)
\( t_{12} = S + 33 \)

8. \( a = \frac{2}{3}, d = -\frac{1}{4}, \text{find } t_9 \)
   - \( n = 9 \)
   - \( n = 9, \text{ find } t_9 \)
   - \( t_9 = \frac{2}{3}(9-1)(-\frac{1}{4}) \)
   - \( t_9 = \frac{2}{3} \cdot 8 \cdot -\frac{1}{4} \)
   - \( t_9 = \frac{8}{3} \)

\( t_9 = \frac{8}{3} + \frac{4}{3} = \frac{12}{3} = t_9 \)

9. \( a = -\frac{3}{4}, d = \frac{1}{2}, \text{find } t_{10} \)
   - \( n = 10 \)
   - \( n = 10, \text{ find } t_{10} \)
   - \( t_{10} = -\frac{3}{4}(10-1)(\frac{1}{2}) \)
   - \( t_{10} = -\frac{3}{4} \cdot 9 \cdot \frac{1}{2} \)
   - \( t_{10} = -\frac{27}{8} \)

\( t_{10} = -\frac{27}{8} + \frac{40}{8} = \frac{13}{4} = t_{10} \)

10. \( a = 2.5, d = -1.25, \text{find } t_{20} \)
    - \( n = 20 \)
    - \( n = 20, \text{ find } t_{20} \)
    - \( t_{20} = 2.5(20-1)(-1.25) \)
    - \( t_{20} = 2.5(19)(-1.25) \)
    - \( t_{20} = -21.25 \)

\( t_{20} = 2.5 + 19(-1.25) \)
\( t_{20} = -21.25 \)

11. \( a = -0.75, d = 0.05, \text{find } t_{40} \)
    - \( n = 40 \)
    - \( n = 40, \text{ find } t_{40} \)
    - \( t_{40} = -0.75 + (40-1)(0.05) \)
    - \( t_{40} = -0.75 + 1.95 \)
    - \( t_{40} = 1.2 \)

\( t_{40} = -0.75 + 1.95 \)
\( t_{40} = 1.2 \)

12. \( a = -\frac{7}{4}, d = -\frac{2}{3}, \text{find } t_{37} \)
    - \( n = 37 \)
    - \( n = 37, \text{ find } t_{37} \)
    - \( t_{37} = -\frac{7}{4} + (37-1)(-\frac{2}{3}) \)
    - \( t_{37} = -\frac{7}{4} + 36(-\frac{2}{3}) \)
    - \( t_{37} = -\frac{7}{4} - 24 \)
    - \( t_{37} = -\frac{7}{4} - 24 = -25.75 = t_{37} \)

\( t_{37} = -\frac{7}{4} + 36(-\frac{2}{3}) \)
\( t_{37} = -\frac{7}{4} - 24 = -25.75 = t_{37} \)
Find the number of terms in each arithmetic sequence

13. \( a = 6, d = -3, t_n = -30 \)  \( n = ? \)

\[
-30 = 6 + (n-1)(-3)
\]
\[
-30 = 6 - 3n + 3
\]
\[
-33 = -3n
\]
\[
n = 11
\]

14. \( a = -3, d = 5, t_n = 82 \)

\[
82 = -3 + (n-1)(5)
\]
\[
85 = 5n - 3
\]
\[
88 = 5n
\]
\[
n = 17.6
\]

15. \( a = 0.6, d = 0.2, t_n = 9.2 \)

\[
9.2 = 0.6 + (n-1)(0.2)
\]
\[
8.6 = 0.2n + 0.6
\]
\[
8 = 0.2n
\]
\[
n = 40
\]

16. \( a = -0.3, d = -2.3, t_n = -39.4 \)

\[
-39.4 = -0.3 + (n-1)(-2.3)
\]
\[
-39.1 = -2.3n + 0.3
\]
\[
-39.4 = -2.3n
\]
\[
n = 18
\]

Find the first term in the arithmetic sequence

17. \(-1, 4, 9, \ldots, 159 \)

\[
t_n = n^2 - 2
\]
\[
159 = n^2 - 2
\]
\[
n^2 = 161
\]
\[
n = 13
\]

18. \(23, 20, 17, \ldots, -100 \)

\[
t_n = a + (n-1)d
\]
\[
-100 = 23 + (n-1)(-3)
\]
\[
-123 = -3n + 23
\]
\[
-146 = -3n
\]
\[
n = 48.67
\]
Section 2.3 – Practice Problems

Find the sum of the arithmetic series

1. \(3 + 5 + 7 + \cdots + (2n + 1)\)
   \[d = 2, \quad a = 3, \quad S_n = \frac{n}{2} \left( t_1 + t_n \right)\]
   \[S_n = \frac{n}{2} \left( 3 + (2n + 1) \right) = \frac{n^2 + 4n}{2}\]
   \[n = \frac{n^2 + 2n}{2}\]

2. \(2 + 5 + 8 + \cdots + 77\)
   \[a = 2, \quad d = 3, \quad S_n = \frac{n}{2} (2 + 77) = \frac{79n}{2}\]
   \[3/58 = \frac{79n}{2}\]

3. \((-41) + (-35) + (-29) + \cdots + 541\)
   \[a = -41, \quad d = 6, \quad S_n = \frac{n}{2} \left( S_{24} = 2450 \right)\]
   \[S_n = \frac{n^2}{2} \left( -41 + 541 \right)\]

4. \(-1 + 2 + 5 + \cdots + (3n - 4)\)
   \[a = -1, \quad d = 3, \quad S_n = \frac{n}{2} \left( -1 + (3n - 4) \right)\]
   \[S_n = \frac{n}{2} (3n - 5)\]

5. \(5 + 9 + 13 + \cdots + 97\)
   \[a = 5, \quad d = 4, \quad S_n = \frac{n}{2} \left( 5 + 97 \right)\]
   \[S_n = \frac{n}{2} (102)\]
   \[39 = 9\]

6. \(2\sqrt{5} + 6\sqrt{5} + 10\sqrt{5} + \cdots + 50\sqrt{5}\)

7. \(39 + 33 + 27 + \cdots + (-15)\)
   \[a = 39, \quad d = -6, \quad S_n = \frac{n}{2} \left( 39 + (-15) \right) = \frac{n}{2} \left( 24 \right)\]
   \[S_n = \frac{n}{2} \left( 24 \right)\]

8. \(23 + 19 + 15 + \cdots + (-305)\)
   \[a = 23, \quad d = -4, \quad S_n = \frac{n}{2} \left( 23 + (-305) \right)\]
   \[S_n = \frac{n}{2} \left( -282 \right)\]
9. $S_{20}$, if $a_1 = 8, a_{20} = 65$

   \[
   a = 8, \quad d = 3, \quad l = 65.
   \]

   \[
   S_{20} = \frac{n}{2} (2a + (n-1)d)
   = \frac{20}{2} (2(8) + (20-1)3)
   = 10(73)
   = 730.
   \]

10. $S_{21}$, if $a_1 = 8, a_{20} = 65$

   \[
   a = 8, \quad l = 65,
   \]

   \[
   S_{21} = \frac{n}{2} (a_1 + a_{21})
   = \frac{21}{2} (8 + 65)
   = 10.5 \cdot 73
   = 764.5.
   \]

11. $S_{56}$, if $a_{56} = 13, d = -9$

   \[
   a = 56, \quad d = -9, \quad a_1 = ?.
   \]

12. $n$ if $S_n = 180, a_1 = 4, a_n = 16$

   \[
   S_n = \frac{n}{2} (2a + (n-1)d)
   = 180,
   \]

   \[
   a = 4, \quad a_n = 16, \quad h = ?.
   \]

13. $d$, if $S_{40} = 680, a_1 = 11$

14. $S_{62}$, if $a_1 = 10, d = 3$

   \[
   S_{62} = \frac{n}{2} (2a + (n-1)d)
   = \frac{62}{2} (10 + (62-1)3)
   = 31 \cdot 37
   = 1147.
   \]

15. $S_{19}$, if $d = 4, a_{19} = 17$

   \[
   S_{19} = 19, \quad d = 4, \quad a_{19} = 17.
   \]

16. $S_{40}$, if $d = -3, a_{40} = 65$
Section 2.4 – Slope and Rate of Change

Slope

- The slope of a linear equation describes the steepness and direction of a line
- As a line is traced from left to right the slope is the vertical change relative to the horizontal change
- There are 4 types of slope

![Slope Types Diagram]

NOTE: USE TIME AND MONEY TO EXPLAIN SLOPE

Finding slope from a graph

Slope \( (m) \)

\[
m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}
\]

Example of positive slope

Slope of segment \( AB = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{6}{4} = \frac{3}{2} \)

Slope of segment \( AC = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{9}{6} = \frac{3}{2} \)

Slope of segment \( BC = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{3}{2} \)

- It doesn’t matter where you start measuring, the slope of a straight line is constant between any two points you pick
- The Slope does not change!
Example of negative slope

Slope of segment DE = \( \frac{\text{vertical change}}{\text{horizontal change}} = -\frac{3}{4} \)

Slope of segment DF = \( \frac{\text{vertical change}}{\text{horizontal change}} = -\frac{6}{8} = -\frac{3}{4} \)

Slope of segment EF = \( \frac{\text{vertical change}}{\text{horizontal change}} = -\frac{3}{4} \)

- It doesn’t matter where you start measuring, the slope of a straight line is constant between any two points you pick.
- The Slope does not change!

Finding Slope from Ordered Pairs

Slope Formula

The slope, \( m \), of a line segment between two points \((x_1, y_1)\) and \((x_2, y_2)\) is

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

\( \star \text{assign your x and y's} \)

Example of positive slope

Slope of segment AB = \( \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 2}{-2 - 2} = -\frac{6}{-4} = \frac{3}{2} \)

Slope of segment AC = \( \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 5}{-2 - 4} = -\frac{9}{-6} = \frac{3}{2} \)

Slope of segment BC = \( \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 5}{2 - 4} = -\frac{3}{-2} = \frac{3}{2} \)

- The order you pick the points will not change the outcome. The Slope will be the same either way.
- Switch the order and give it a try!
Example of negative slope

Slope of segment DE = \( \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(−2)}{−3−1} = \frac{3}{−4} = −\frac{3}{4} \)

Slope of segment DF = \( \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(−5)}{−3−5} = \frac{6}{−8} = −\frac{3}{4} \)

Slope of segment EF = \( \frac{y_2 - y_1}{x_2 - x_1} = \frac{−2−(−5)}{1−5} = \frac{3}{−4} = −\frac{3}{4} \)

- The order you pick the points will not change the outcome. The slope will be the same either way.
- Switch the points and give it a try!

Lines with Zero Slope and Undefined Slope

- If two different points have the same \( y \) - value, the line (or line segment) joining the two points is horizontal. (Think \( h \) looks like an upside down \( y \))

**Example of Zero Slope**

\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0 \]

- If two different points have the same \( x \) - value, the line (or line segment) joining the two points is vertical. (Think \( x \) looks like two back to back \( v \)'s)

**Example of Zero Slope**

\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{0} = Undefined \]
Rate of Change

- The Greek letter Delta (Δ) is used to represent change.
- We use Rates of Changes to help compare quantities with different units.
- The formula for Rate of Change is: change in y over change in x.

\[
\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

- Does this look familiar?
- What is the equation for Slope?

Examples of Rates of Change:

1. Kilometers per hour: \( \frac{km}{hr} \) or \( \frac{km}{h} \)
2. Miles per gallon: \( \frac{miles}{gal} \) or \( \frac{mi}{gal} \)
3. Dollars per hour: \( \frac{\$}{hr} \) or \( \frac{dollars}{hour} \)

4. If the city of Surrey grew by 120 000 people over a five year period.
   It has a rate of change of: \( \frac{120,000 \text{ people}}{5 \text{ years}} = 24,000/\text{yr} \)

5. If a person runs the 400m race in 56 seconds, they run at a rate of:
   \( \frac{400m}{56 \text{ sec}} = 7.14 \text{ meters/second} \)

- Rates of Change are just the slope relationship of two variables
- The variable on the y-axis is the independent variable
- The variable on the x-axis is the dependent variable (Usually: TIME)
Example:
Paul rents a car with a full gas tank. The odometer read 86,347 km. Paul used the car for 3 days. When he returned it the odometer read was 86,721 km and it needed 63 litres to fill up. The cost of renting the car was $96 plus gas which cost 90 cents per litre.

a) Determine the rate of gas consumption for the car.
b) Determine the average rate of travel per day.
c) Determine the cost of renting the car per day.

Solution:
\[
\frac{\Delta y}{\Delta x} = \frac{(86,721-86,347) \text{ km}}{(63-0) \text{ litres}} = 5.94 \text{ km/litre}
\]
\[
\frac{\Delta y}{\Delta x} = \frac{(86,721-86,347) \text{ km}}{(3-0) \text{ days}} = 124.7 \text{ km/day}
\]
\[
\frac{\Delta y}{\Delta x} = \frac{(96-0) \text{ $}}{(3-0) \text{ days}} \frac{63(90)}{3} = 50.90/\text{day}
\]

- Rates of Change can be visualized using graphs. As mentioned the denominator quantity is generally placed on the x-axis, the numerator value is placed on the y-axis.

Example: Between 2000 and 2010, the cost of a 42” LCD TV dropped from $4600 to $1200. Graph this result and determine the average drop in price per year.

Solution: LCD TVs have been dropping at an average rate of $340 per year.
Example: Most cars depreciate as they age. A car costing $30 000 will have a value of $2500 at the end of 10 years.

a) Write a formula for its value \( V \), when it is \( t \) years old. \( 0 \leq t \leq 10 \)
b) Draw the graph of this linear function
c) Determine the car’s value after 4.5 years
d) When is the car’s value between $12 000 and $15 000?
e) How much value does the car lose every 2.5 years?
f) What is the rate of change of the car’s value with respect to time?

Solution:

\[
\text{a) Slope: } \frac{\Delta y}{\Delta x} = \frac{30000 - 2500}{0 - 10} = \frac{27500}{-10} = -2750/\text{yr.} \quad \text{Therefore } V = 30000 - 2750t
\]

\[
y = mx + b
\]

\[
m = \frac{25000 - 30000}{10 - 0} = \frac{-5000}{10} = \frac{-27500}{10} = -2750 \text{ per year}
\]

c) \( V = 30000 - 2750(4.5) = $17 625 \)

d) \( V = 30000 - 2750(t) = 12000 \rightarrow 2750t = 18000 \rightarrow t = 6.5 \text{ years} \)

\[V = 30000 - 2750(t) = 15000 \rightarrow 2750t = 15000 \rightarrow t = 5.5 \text{ years} \]

e) Since it loses $2750 per year, $2750 \times 2.5 \text{ years} = $6875 \text{ lost every } 2.5 \text{ years}

f) The rate of change of value is the slope \( = -2750 \text{ per year} \)
Example:
Georgia sells computers. She is paid a basic monthly salary of $1500, plus $400 for every five computers she sells.

a) Write a formula for Georgia's monthly wage.
b) How many computers must be sold for Georgia to make at least $3440 in one month?
c) Determine Georgia's wage in a month when she sells 60 computers
d) What is the rate of change of Georgia's wage with respect to the number of computers sold?

Solution:
a) \[ \frac{\Delta y}{\Delta x} = \frac{400 - 0}{5 - 0} = 80 \Rightarrow \text{So,} \quad W = 1500 + 80x \]

b) \[ W = 1500 + 80x = 3440 \Rightarrow 80x = 1940 \Rightarrow x = 24.25 \]

Georgia must sell 25 computers

c) \[ W = 1500 + 80(60) = 6300 \]

d) The rate of change is the slope, $80$ per computer

Example:
In the morning, Anna types nine pages in 45 minutes. After lunch, she typed 18 pages in 1 hour and 20 minutes. If the pages typed were approximately the same length, did she type faster in the morning or after lunch?

Solution:
Determine the rate in the morning

\[ \frac{9 \text{ pages}}{45 \text{ minutes}} = 0.2 \text{ pages/min} \]

Determine the rate after lunch

\[ \frac{15 \text{ pages}}{80 \text{ minutes}} = 0.1875 \text{ pages/min} \]

Therefore, Anna typed faster in the morning.
Section 2.4 – Practice Problems

Fill in the blank with the appropriate word

1. The run between two points on a coordinate system refers to change in the \(x\) variable
2. The rise between two points on a coordinate system refers to change in the \(y\) variable
3. The letter \(m\) is used to indicate the slope of a line
4. The formula for finding the slope of a line is \(\frac{y^2 - y^1}{x^2 - x^1}\)
5. The slope of a vertical line is \(\text{undefined}\)
6. The slope of a horizontal line is \(\text{zero}\)
7. A positive has both the \(x\)-coordinates and \(y\)-coordinates increasing
8. A negative has both the \(x\)-coordinates and \(y\)-coordinates decreasing
9. Slope represents a rate of change

10. Match the column on the left with the column on the right

   a) rise \(\rightarrow\) i) \(x = 3\)
   b) run \(\rightarrow\) ii) difference in \(x\)
   c) slope \(\rightarrow\) iii) \(\frac{\text{difference in } y}{\text{difference in } x}\)
   d) vertical line \(\rightarrow\) iv) difference in \(y\)
   e) horizontal line \(\rightarrow\) v) \(y = -1\)

Determine if the slope is positive, negative, zero, or undefined.

11.

12.

13.

14.

Adrian Herlaar, School District 61

www.mrherlaar.weebly.com
Determine the slope of the line

15. \[ m = \frac{3}{2} \]

16. \[ m = \frac{1}{5} \]

17. \[ \frac{2.5}{3.0} \quad \frac{1}{5} \quad m = 4 \]

18. \[ m = \frac{1}{4} \]

19. \[ \text{zero} \]

20. \[ \text{undefined} \]

Find the slope from the points provided

21. \((2, 3)\) and \((6, 9)\) \[ \frac{9 - 3}{6 - 2} = \frac{6}{4} = \frac{3}{2} \]

22. \((3, 2)\) and \((7, 10)\) \[ \frac{10 - 2}{7 - 3} = \frac{8}{4} = 2 \]

23. \((-1, 5)\) and \((4, 1)\) \[ \frac{1 - 5}{4 - (-1)} = \frac{-4}{5} \]

24. \((2, 2)\) and \((2, -2)\) \[ \text{undefined} \]

25. \((2, -1)\) and \((-5, -1)\)

26. \((-3, 1)\) and \((6, 8)\) \[ \text{undefined} \]
27. A long distance runner passes the 24km mark of a race in 1hr 20 min, and passes the 42km mark 1 hour later. Assuming a constant rate, find the speed of the long distance runner in km/hr.

\[
\frac{42 - 24}{1 - 0} = \frac{18}{1} = 18 \text{ km/h}
\]

28. A plane at an altitude of 20,000 feet starts to descend for landing after flying for six hours. The entire flight time was 6 hours and 40 minutes. Determine the average rate of descent of the plane.

\[
\begin{align*}
(60,20,000) & \quad (400,0) \\
\frac{20,000}{40} & = \frac{500}{1} \\
\Rightarrow & \quad 500 \text{ ft/min.}
\end{align*}
\]

29. As a window washer begins work on a high rise, one-third of the windows were already clean. Eight hours later, three-quarters of all the windows are clean. Calculate the window washer's cleaning rate.

\[
\frac{3 \cdot \frac{1}{3}}{8 - 0} = \frac{1}{4} \text{ window/hour}
\]

30. A five foot long treadmill rises six inches to make an incline for running. What is the slope of the treadmill?

\[\alpha + \beta = \gamma\]

Class example:

\[
\begin{align*}
\text{tan}\theta &= \frac{5 \text{ ft}}{1 \text{ ft}} \\
\text{tan}\beta &= \frac{6 \text{ in}}{1 \text{ ft}} \\
\text{tan}\gamma &= \frac{100 - 10}{100}
\end{align*}
\]
Section 2.5 – Graphing Linear Functions

Points and Slope Relationships

- If we know the slope of a line, and a point on it, we can graph the line.
- Using the slope we can also find any other point on the line

Example:

Graph a line with a slope of 2, going through the point \((-1, -4)\).

Solution:

- Plot the point you start with: \((-1, -4)\)
- Trace your slope to the next point
- Continue that tracing until satisfied
- Connect the points

Slope is: \(\frac{\text{Rise}}{\text{Run}} = \frac{2}{1}\) so, UP 2, RIGHT 1 from the starting point

But...

\[2 = \frac{-2}{-1} = \frac{2}{1}\] so, you can go DOWN 2, LEFT 1, from any point too

Example:

Find a point in quadrant IV on a line with slope \(-\frac{1}{3}\) going through the point \((-5, 2)\).

Solution:

- Plot your first point: \((-5, 2)\)
- Trace your slope of: \(-\frac{1}{3}\)
- Remember that:

\[\text{Slope} = \frac{-1}{3} = \frac{-1}{3} = \frac{1}{-3}\]

So you can trace, DOWN 1, RIGHT 3 or...

You can trace, UP 1, LEFT 3

Which will get you where you need to go?

- Look for a point where the line intercepts a cross section visually
- One possible answer is \((4, -1)\)
- There are an infinite number of answers
Example:
Determine the slope of the graph.

Solution:
- Two points of the graph that have clear x and y
  integer coordinates are: (−2, 4) and (3, 1).
- \[ m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 4}{3 - (-2)} = \frac{-3}{5} = -\frac{3}{5} \]
- So the Slope is: \[ -\frac{3}{5} \]

Intercepts
- The point where the line crosses the y-axis is called the y-intercept
- The point where the line crosses the x-axis is called the x-intercept
- A linear relation can have one intercept, two intercepts or an infinite number of intercepts

\[ x-\text{intercept} \text{ and } y-\text{intercept} \]

The x-intercept of a line is the point \((a, 0)\) where the line intersects the x-axis

The y-intercept of a line is the point \((0, b)\) where the line intersects the y-axis

Horizontal or Vertical
Lines have 1 INTERCEPT

Diagonal Lines have 2
INTERCEPT2

Horizontal or Vertical
Lines on the axes have
INFINITE INTERCEPTS
Example:
Determine the $x$ and $y$ - intercepts of the linear equation with slope $\frac{2}{3}$, going through $(-6, -2)$

Solution:
- Plot your starting point and trace your slope
- The $x$ - intercept is $(-3, 0)$
- The $y$ - intercept is $(0, 2)$

*At this point we can only eyeball the intercepts, in the next section we will see an equation that helps us find them even if they are not easily discernable*

Example:
Determine the slope of a line with $x$ - intercept $(-3, 0)$ and $y$ - intercept $(0, -4)$.

Solution:
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-4)}{-3 - 0} = -\frac{4}{3}$$

Parallel Lines
- We can determine if two lines are parallel or perpendicular using slopes.
- Parallel Lines are lines on a graph (in a coordinate system) that never intersect
- They have identical slopes because they rise or fall at the same rates

- The only difference between the two lines in their $y$ - intercepts
Perpendicular Lines

- Perpendicular lines are lines that form right angles when they intersect.
- If the slope of one line is \( \frac{a}{b} \), the slope of the perpendicular line is \( -\frac{b}{a} \).
- These are called: Negative Reciprocals (Refliprocal to remember!)
- The product of the slopes of perpendicular lines is \(-1\).

\[
m_2 = -\frac{1}{2} \quad m_1 = 2
\]

\[
m_1 \cdot m_2 = 2 \cdot \left(-\frac{1}{2}\right) = -1
\]

Example:

Determine if the line through the first pair of points is parallel to, perpendicular to, or neither parallel nor perpendicular to the line through the second pair of points.

a) \((-4,1) \text{ and } (3,5); \ (1,-3) \text{ and } (15,-11)\)
b) \((-4,1) \text{ and } (3,5); \ (1,-11) \text{ and } (15,-3)\)
c) \((-4,1) \text{ and } (3,5); \ (-13,10) \text{ and } (-9,3)\)

Solution:

a) \[
m_1 = \frac{y_2-y_1}{x_2-x_1} = \frac{1-5}{-4-3} = \frac{-4}{-7} = \frac{4}{7}
\]

\[
m_2 = \frac{y_2-y_1}{x_2-x_1} = \frac{-3-(-11)}{1-15} = \frac{8}{-14} = -\frac{4}{7}
\]

\[m_1 \neq m_2 \text{ and } m_1 \cdot m_2 \neq -1, \text{ so the lines are neither parallel nor perpendicular}\]

b) \[
m_1 = \frac{y_2-y_1}{x_2-x_1} = \frac{1-5}{-4-3} = \frac{-4}{-7} = \frac{4}{7}
\]

\[
m_2 = \frac{y_2-y_1}{x_2-x_1} = \frac{-11-(-3)}{1-15} = \frac{-8}{-14} = \frac{4}{7}
\]

\[m_1 = m_2, \text{ therefore the lines are parallel}\]
Applications of Linear Relations and Graphing Data

- These types of questions focus on intercepts, slope, domain and range

Example:

A TV repair company charges a fixed amount, plus an hourly rate for a service call. A two hour service call is $80, and a four hour service call is $140.

a) Write the equation that shows how the total cost, $T$, depends on the number of hours, $h$, and the fixed cost, $C$. Use $R$ for the hourly rate.

b) Find the hourly rate.

c) Find the fixed amount.

d) Find the domain and range.

Solution:

a) $T = Rh + C$

b) The hourly rate is the slope.

$$R = \frac{\Delta T}{\Delta h} = \frac{\text{change in total cost}}{\text{change in hours}} = \frac{T_4 - T_2}{h_4 - h_2} = \frac{140 - 80}{4 - 2} = \frac{60}{2} = 30/\text{hr}$$

The fixed cost is $20.

For 2 hours, $T$, is $80, therefore $C = T - 30h = 80 - 30(2) = 20$

For 4 hours, $T$, is $140, therefore $C = T - 30h = 140 - 30(4) = 20$

d) The domain is \{0, 1, 2, 3, ...\}  
The range is \{20, 50, 80, 110, 140, ...\}

We do not connect the dots because the company charges per hour not minute or second. It is not a continuous relationship, it is discrete.
Example:

An antique dresser increases in value $50 per year. The dresser is worth $600 now.

a) Write the equation for the current worth of the dresser, $C$, depending on the years, $t$.

b) What price was paid for the dresser if it was bought three years ago?

c) What will the value of the dresser be in five years?

d) Determine the domain and range.

You can see the Linear Pattern but again we do not connect the dots because the value increases per year not month or day. It is not a continuous relationship, it is discrete.

Solution:

a) $C = 600 + 50t$

b) $C = 600 + 50t = 600 + 50(-3) = 600 - 150 = 450$

The dresser initially cost $450.

c) $C = 600 + 50t \rightarrow C = 600 + 50(5) \rightarrow C = 600 + 250 = 850$

The dresser will be worth $850 in five years

d) The domain is $\{-3, -2, -1, 0, 1, 2, 3, ...\}$

The range is $\{450, 500, 550, 600, 650, ...\}$
Section 2.5 – Practice Questions

Graph the line that passes through the given point and has the given slope

1. \((0, 2); m = 1\)

2. \((-3, 1); m = 2\)

3. \((-2, -2); m = -\frac{3}{4}\)

4. \((-3, -1); m = \frac{3}{2}\)

5. \((6, -4); m = -\frac{3}{5}\)

6. \((3, -5); m = -\frac{2}{3}\)
7. \((-4, 1); m = 0\)

8. \((-2, -5); m = \text{undefined}\)

Graph the line with the given slope and intercept

9. \(x\ -\ text{intercept:}\ (-2, 0); m = 1\)

10. \(y\ -\ text{intercept:}\ (0, -3); m = \frac{-2}{1}\)

11. \(x\ -\ text{intercept:}\ (3, 0); m = \frac{-3}{4}\)

12. \(y\ -\ text{intercept:}\ (0, -1); m = \frac{3}{2}\)
Determine the slope of the line with the given $x$ and $y$ intercepts

13. $(2, 0), (0, 2)$
   \[
   \frac{0-2}{2-0} = \frac{-2}{2} = -1
   \]

14. $(-2, 0), (0, -2)$
   \[
   \frac{0-(-2)}{0-(-2)} = \frac{2}{2} = 1
   \]

15. $(2, 0), (0, -2)$

16. $(-2, 0), (0, 2)$

17. $(0, 3), (0, 0)$
   \[
   \text{undefined}
   \]

18. $(-4, 0), (0, 0)$

Determine whether the line passing through the first pair of points is parallel, perpendicular, or neither to the line passing through the second pair of points

19. $(3, 2)$ and $(1, 4)$; $(-1, -2)$ and $(-3, -4)$
   \[
   \frac{4-2}{1-3} = \frac{2}{-2} = -1
   \]

20. $(5, 6)$ and $(7, 8)$; $(-5, -6)$ and $(-7, -8)$
   \[
   \frac{8-6}{7-5} = \frac{2}{2} = 1
   \]

21. $(0, 4)$ and $(-1, 2)$; $(-3, 5)$ and $(1, 7)$
   \[
   \frac{2-4}{-1-0} = \frac{-2}{-1} = 2
   \]

22. $(2, 3)$ and $(3, 0)$; $(-2, -5)$ and $(1, -6)$
   \[
   \frac{0-3}{3-2} = \frac{-3}{1} = -3
   \]

23. $(3, 5)$ and $(-2, 5)$; $(1, 4)$ and $(1, -2)$
   \[
   \frac{5-5}{-2-3} = \frac{0}{5}
   \]

24. $(4, -3)$ and $(-2, -1)$; $(10, -1)$ and $(1, -4)$
   \[
   \frac{-3-(-1)}{1-0} = \frac{-2}{-1} = 2
   \]

25. $(a, b)$ and $(b, a)$; $(c, d)$ and $(d, c)$

26. $(a, b)$ and $(b, a)$; $(-c, d)$ and $(-d, c)$

27. Show that the points $A(-3, 0), B(1, 2),$ and $C(3, -2)$ are the vertices of a right triangle
28. The line through \((r, -6)\) and \((-2, -1)\) is perpendicular to a line with slope \(-2\). Find \(r\).

29. A taxi driver charges a passenger \$21.50 \text{ to travel } 15\text{km}, \text{ and charged another passenger } \$37.10 \text{ to travel } 28\text{km}.
   a) Find the cost per km
   
   b) Write the equation that shows how the total cost, \(T\), depends on the number of kilometers, \(K\), plus a fixed amount
   
   c) How far can a person travel for \$53.90
   
   d) Determine the Domain and Range

30. A four year old car is worth \$27,600, and will be worth \$4200 \text{ 10 years from now.}
   a) Find the yearly depreciation of the car
   
   b) Write the equation that shows the value of the car, \(V\), depends on the new cost of the car, \(N\), and how many years old it is, \(Y\).
   
   c) Find the new price of the car
   
   d) Determine the Domain and Range
<table>
<thead>
<tr>
<th>Section 2.1</th>
<th>Section 2.2</th>
<th>Section 2.3</th>
<th>Section 2.4</th>
<th>Section 2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. IV</td>
<td>1. c</td>
<td>1. $-1, 2, 7, 14$</td>
<td>1. $x$</td>
<td>1. See Written</td>
</tr>
<tr>
<td>2. I</td>
<td>2. $d$</td>
<td>2. $\frac{3}{4}, \frac{5}{8}, \frac{7}{8}$</td>
<td>2. $y$</td>
<td>2. See Written</td>
</tr>
<tr>
<td>3. II</td>
<td>3. Yes</td>
<td>3. $1, -4, 9, -16$</td>
<td>3. $m$</td>
<td>3. See Written</td>
</tr>
<tr>
<td>4. III</td>
<td>4. Yes</td>
<td>4. $\frac{2}{3}, -\frac{4}{5}$</td>
<td>4. $\frac{y_2 - y_1}{x_2 - x_1}$</td>
<td>4. See Written</td>
</tr>
<tr>
<td>5. No Quadrant</td>
<td>5. No</td>
<td>5. $9, 3.81$</td>
<td>5. Undefined</td>
<td>5. See Written</td>
</tr>
<tr>
<td>6. NO Quadrant</td>
<td>6. No</td>
<td>6. $1.5, 3.17$</td>
<td>6. 0</td>
<td>6. See Written</td>
</tr>
<tr>
<td>7. See Written</td>
<td>7. Yes</td>
<td>7. $2, 1, \frac{8}{9}, 1$</td>
<td>7. Positive</td>
<td>7. See Written</td>
</tr>
<tr>
<td>8. a</td>
<td>8. No</td>
<td>8. $\frac{2}{4}, \frac{8}{16}$</td>
<td>8. Negative</td>
<td>8. See Written</td>
</tr>
<tr>
<td>10. Function</td>
<td>10. No</td>
<td>10. $t_{32} = 38$</td>
<td>10. a) $iv$</td>
<td>10. See Written</td>
</tr>
<tr>
<td>11. Function</td>
<td>11. See Written</td>
<td>11. $t_{10} = \frac{125}{4}$</td>
<td>b) $ii$</td>
<td>11. See Written</td>
</tr>
<tr>
<td>12. Not a Function</td>
<td>12. See Written</td>
<td>12. $t_{20} = -21.25$</td>
<td>c) $iii$</td>
<td>12. See Written</td>
</tr>
<tr>
<td>13. Function</td>
<td>13. See Written</td>
<td>13. $t_{40} = 1.2$</td>
<td>d) $i$</td>
<td>13. $-1$</td>
</tr>
<tr>
<td>14. Function</td>
<td>14. See Written</td>
<td>14. $t_{37} = -25.75$</td>
<td>e) $v$</td>
<td>14. $-1$</td>
</tr>
<tr>
<td>15. Function</td>
<td>15. See Written</td>
<td>15. $n = 13$</td>
<td>11. Positive</td>
<td>15. 1</td>
</tr>
<tr>
<td>16. 1 – 1</td>
<td>16. See Written</td>
<td>16. $n = 18$</td>
<td>12. 0</td>
<td>16. 1</td>
</tr>
<tr>
<td>20. See Written</td>
<td>20. See Written</td>
<td>20. $n = 42$</td>
<td>16. $m = \frac{5}{3}$</td>
<td>20. Parallel</td>
</tr>
<tr>
<td>22. See Written</td>
<td>22. See Written</td>
<td>22. $a = -4$</td>
<td>18. $m = -\frac{1}{4}$</td>
<td>22. Neither</td>
</tr>
<tr>
<td>23. See Written</td>
<td>23. See Written</td>
<td>23. $a = 47$</td>
<td>19. $m = 0$</td>
<td>23. Perp</td>
</tr>
<tr>
<td>24. $a = 40\frac{1}{3}$</td>
<td>24. See Written</td>
<td>24. $a = 15$</td>
<td>20. Undefined</td>
<td>24. Neither</td>
</tr>
<tr>
<td>25. $a = 21$</td>
<td>25. See Written</td>
<td>25. $a = 21$</td>
<td>21. $m = \frac{3}{2}$</td>
<td>25. Parallel</td>
</tr>
<tr>
<td>26. See Written</td>
<td>26. See Written</td>
<td>26. $a = 21$</td>
<td>22. $m = 2$</td>
<td>26. Perp</td>
</tr>
<tr>
<td>27. See Written</td>
<td>27. See Written</td>
<td>27. $a = 21$</td>
<td>23. $m = -\frac{4}{5}$</td>
<td>27. See Written</td>
</tr>
<tr>
<td>29. See Written</td>
<td>29. See Written</td>
<td>29. $a = 21$</td>
<td>25. $m = 0$</td>
<td>29. a) $$1.20/km</td>
</tr>
<tr>
<td>30. See Written</td>
<td>30. See Written</td>
<td>30. $a = 21$</td>
<td>26. $m = \frac{7}{9}$</td>
<td>b) $T = 1.20k + 3.50$</td>
</tr>
<tr>
<td>31. See Written</td>
<td>31. See Written</td>
<td>31. $a = 21$</td>
<td>27. $18 \text{ km/hr}$</td>
<td>c) $k = 42$ km</td>
</tr>
<tr>
<td>32. See Written</td>
<td>32. See Written</td>
<td>32. $a = 21$</td>
<td>28. $500 \text{ ft/min}$</td>
<td>d) $D: {0, 1, 2, 3, \ldots}$</td>
</tr>
<tr>
<td>33. See Written</td>
<td>33. See Written</td>
<td>33. $a = 21$</td>
<td>29. $\frac{5}{96}$</td>
<td>$R: {3.50, 4.70, 5.90, \ldots}$</td>
</tr>
<tr>
<td>34. See Written</td>
<td>34. See Written</td>
<td>34. $a = 21$</td>
<td>30.</td>
<td>30. a) $$ -2340/yr</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>b) $V = -2340(y) + N$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>c) $N = $36,960$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>d) $D: {0, 1, 2, 3, \ldots}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$R: {36,960, 34,620, \ldots}$</td>
</tr>
</tbody>
</table>