Section 7.5 – Double-Angle Identities

- The last section we will look at for Pre-Calculus 12 Trigonometry are Double Angle Identities
- We can prove and derive these identities directly from the Sum and Difference Identities

Consider Sine…

$$\sin(A+B) = \sin A \cos B + \cos B \sin A$$

- Now substitute the B for another A

$$\sin(A+A) = \sin A \cos A + \cos A \sin A$$

- Therefore,

$$\mathbf{\sin 2A = 2\sin A \cos A}$$

Since the angle given is not just A, but $2A$. It is literally a Double Angle!

We can use the same approach for Cosine…

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

- Now substitute the B for A again.

$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$

- Therefore,

$$\mathbf{\cos 2A = \cos^2 A - \sin^2 A}$$

Cosine Double Angles however, in conjunction with our Pythagorean Identity $\sin^2 A + \cos^2 A = 1$ has a couple of other possible identities.

Since, $\sin^2 A = 1 - \cos^2 A$ we can substitute $1 - \cos^2 A$ into the Double Angle Equation for $\sin^2 A$

$$\mathbf{\cos 2A = \cos^2 A - \sin^2 A} \quad \rightarrow \quad \cos 2A = \cos^2 A - (1 - \cos^2 A) \quad \rightarrow \quad \mathbf{\cos 2A = 2\cos^2 A - 1}$$

Since, $\cos^2 A = 1 - \sin^2 A$ we can substitute $1 - \sin^2 A$ into the Double Angle Equation for $\cos^2 A$

$$\mathbf{\cos 2A = \cos^2 A - \sin^2 A} \quad \rightarrow \quad \cos 2A = 1 - \sin^2 A - \mathbf{\sin^2 A} \quad \rightarrow \quad \mathbf{\cos 2A = 1 - 2\sin^2 A}$$

Now let's look at our good friend Tangent:

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

Again, we replace B with A

$$\tan(A+A)=\frac{\tan A+\tan A}{1-\tan A\tan A}$$

$$\mathbf{\tan 2A=\frac{2\tan A}{1-\tan^2 A}}$$

This gives us the following Double Angle Identities

Double-Angle Formulas

$\mathbf{\sin 2A = 2\sin A\cos A}$	$\mathbf{\cos 2A=\cos^2 A-\sin^2 A}$ $\mathbf{\cos 2A = 2\cos^2 A - 1}$ $\mathbf{\cos 2A = 1-2\sin^2 A}$	$\mathbf{\tan 2A=\frac{2\tan A}{1-\tan^2 A}}$

Example 1: Solve: $\cos 2x = 2\sin^2 x,\quad 0\le x<2\pi$

Solution 1: You see that we do not have x we have $2x$. In Section 7.3 we painstakingly solved these, but these Double Angle Formulas may help!

When dealing with $\cos 2x$ you need to look at the equation and pick the identity you are going to use strategically.

$$\cos 2x = 2\sin^2 x$$

$$1-2\sin^2 x = 2\sin^2 x$$

$$1 = 4\sin^2 x$$

In this case, it is best to use:

$1-2\sin^2 x$

$$\sin^2 x=\frac{1}{4}\quad\rightarrow\quad \sin x=\pm\frac{1}{2}$$

$$x=\sin^{-1}\left(\frac{1}{2}\right)\quad\rightarrow\quad x=\sin^{-1}\left(-\frac{1}{2}\right)$$

Positive in $Q1$ and $Q2$

$$x=\frac{\pi}{6}\ and\ \frac{5\pi}{6}\quad\rightarrow\quad x=\frac{7\pi}{6}\ and\ \frac{11\pi}{6}$$

Negative in $Q3$ and $Q4$

Solutions are: $\mathbf{\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}}$

Example 2: Use Double Angle Formula to simplify:

a) $12\sin 4x\cos 4x$

b) $4 - 8\cos^2 6x$

c) $\dfrac{4\tan 3x}{1 - \tan^2 3x}$

Solution 2: It is helpful to consider the object of the trigonometric function as one this when it is a product. Use substitution to simplify the way you view it at first.

a) $12\sin\theta\cos\theta = 6(2\sin\theta\cos\theta)$

Let $4x = \theta$

$= 6\sin 2\theta$
$= 6\sin 2(4x)$
$= 6\sin 8x$

Sub $4x$ back in for θ

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b) $4 - 8\cos^2\theta = -4(2\cos^2\theta - 1)$

Let $6x = \theta$

$= -4\cos 2\theta$
$= -4\cos 2(6x)$
$= -4\cos 12x$

Sub $6x$ back in for θ

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c) $\dfrac{4\tan\theta}{1-\tan^2\theta} = 4\left(\dfrac{\tan\theta}{1-\tan^2\theta}\right)$

Let $3x = \theta$

$= 4\tan 2\theta$
$= 4\tan 2(3x)$
$= 4\tan 6x$

Sub $3x$ back in for θ

Example 3: Prove the identity:

$$\frac{\sin 6x}{1+\cos 6x} = \tan 3x$$

Solution 3: $6x = 2(3x)$, so for simplicity let $3x = \theta$. That way we have 2θ

$$\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$$

$$\frac{2\sin\theta\cos\theta}{1+(2\cos^2\theta - 1)} = \frac{\sin 3x}{\cos 3x}$$

$$\frac{2\sin\theta\cos\theta}{2\cos^2\theta} =$$

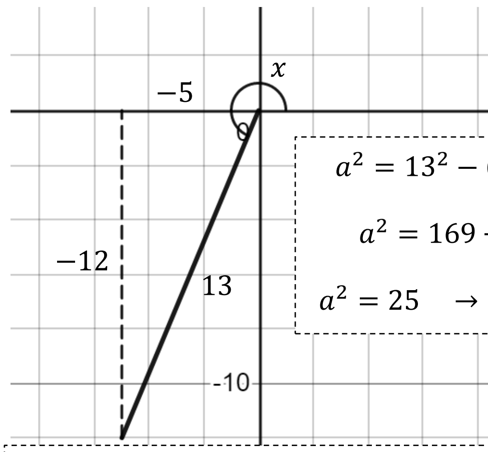
$$\frac{\sin\theta}{\cos\theta} =$$

$$\frac{\sin 3x}{\cos 3x} =$$

Once you have done the hard transformations, sub $3x$ back in for θ

Example 4: Given $\sin x = \frac{12}{13}$ in $Q3$, find $\tan 2x$

Solution 4: Let's look at this graphically to see if we find all the necessary information



$$a^2 = 13^2 - (-12)^2$$

$$a^2 = 169 - 144$$

$$a^2 = 25 \quad \rightarrow \quad a = -5$$

$$\tan x = -\frac{12}{-5} = \frac{12}{5}$$

$$\mathbf{\tan 2x = \frac{2\tan x}{1 - \tan^2 x}}$$

$$\tan 2x = \frac{2\left(\frac{12}{5}\right)}{1 - \left(\frac{12}{5}\right)^2}$$

$$\tan 2x = \frac{\frac{24}{5}}{1 - \frac{144}{25}} = \frac{\frac{24}{5}}{-\frac{119}{25}}$$

$$\mathbf{\tan 2x = \frac{24}{5} \cdot -\frac{25}{119} = -\frac{120}{119}}$$

Power Reducing Identities

- There may come a scenario when we want to reduce the power of a trigonometric identity
- The proofs are derived by algebraically manipulating:

$\cos 2x = 1 - 2\sin^2 x$	$\cos 2x = 2\cos^2 x - 1$	$\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$

- I will leave the manipulation to you, but the result is:

Power-Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad \tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Section 7.5 – Practice Problems

1. Simplify the following expressions

a) $8\sin 5x \cos 5x$

b) $4\sin\frac{x}{2}\cos\frac{x}{2}$

c) $2\sin^2 2x - 2\cos^2 2x$

d) $\frac{8\tan 4x}{1-\tan^2 4x}$

e) $\sec 8x\,(\sin^2 4x - \cos^2 4x)$

f) $2\sin 6x(\cos^2 3x - \sin^2 3x)$

g) $\frac{1}{2}\cot 4x(1 - \tan^2 4x)$

h) $\frac{1}{4}\sec 6x\csc 6x$

i) $4\sin^2\frac{x}{2} - 2$

j) $2\cos^2 8x - 1$

k) $\frac{\sin 6x}{2\sin 3x}$

l) $\sin 4x \csc 2x - 2\cos 2x$

m) $\sin 4x - (\sin 2x + \cos 2x)^2$

n) $\sin^4 3x - \cos^4 3x$

o) $\frac{2}{1-\cos 8x}$

p) $\frac{4}{\tan 3x-\cot 3x}$

2. Solve, $0 \leq x < 2\pi$

a) $\sin 2x + \cos x = 0$

b) $\sin x + \cos 2x = 1$

c) $3\cos 2x + 2\sin^2 x = 2$

d) $\sin 2x = \cot x$

e) $\csc^2 x = 2\sec 2x$

f) $\tan x - \cot x = 2$

g) $\tan 2x + \tan x = 0$

h) $4\sin^2 x = 2 - \cos^2 2x$

i) $\cos 4x + 2\cos^2 2x = 2$

j) $\csc^2 x = 2\sec 2x$

3. Prove the following identities

a) $\cot x - \tan x = \frac{4\cos^2 x - 2}{\sin 2x}$

b) $\frac{\sin 4x - \sin 2x}{\cos 4x + \cos 2x} = \tan x$

c) $\tan 2x = \dfrac{2}{\cot x - \tan x}$

d) $\dfrac{\cot x - \cos x}{1 - \sin x} = \dfrac{\sin 2x}{1 - \cos 2x}$

e) $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$

f) $\dfrac{\cos 2x}{1 - \sin 2x} = \dfrac{1 + \tan x}{1 - \tan x}$

4. If $\sin x = -\frac{3}{5}$ in $Q3$, find:

a) $\sin 2x$

b) $\cos 2x$

c) $\tan 2x$

5. If $\tan x = -3$, in $Q2$, find:

a) $\sin 2x$

b) $\tan 2x$

c) $\sec 2x$

6. Write $\cos 3x$ in terms of $\cos x$

7. Write $\cos 4x$ in terms of $\cos x$

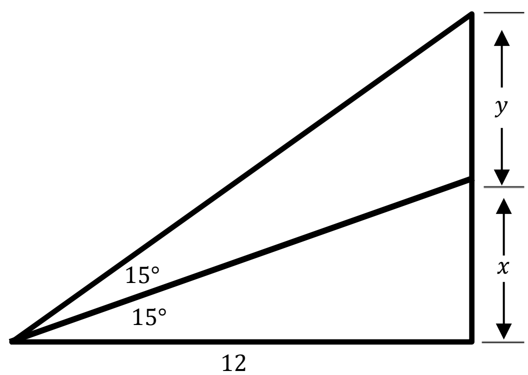
8. Write $\sin 5x$ in terms of $\sin x$

9. Write $\tan^4 x$ in terms of a first power.

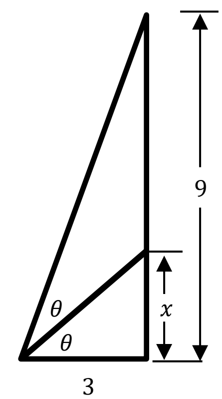
10. Write $\sin^2 x \cos^4 x$ in terms of the first power of Cosine.

11. Write $\sin^4 x + \cos^4 x$ in terms of the first power of Cosine.

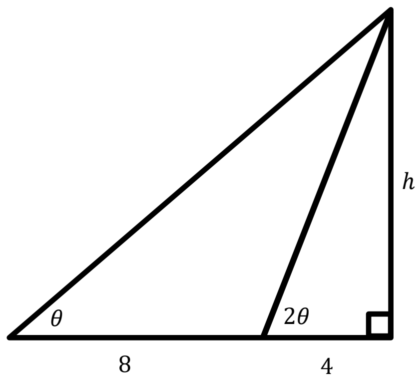
12. Find y



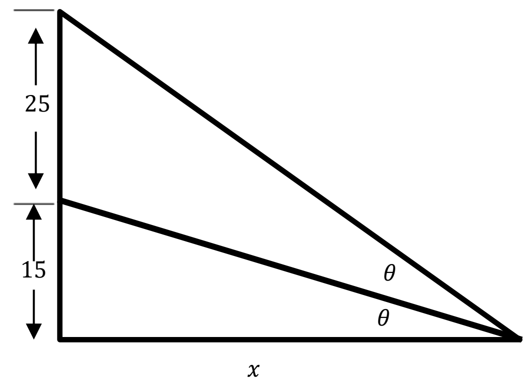
13. Find x



14. Find h



15. Find x



See Website for Detailed Answer Key

Extra Work Space