Section 7.4 – Sum and Difference Identities

- We have a very clear and thorough proof on the website to explain and demonstrate where the following identities are derived from
- For the sake of this course, take these identities at face value

Sum and Difference Identities

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B} \qquad \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

Also, consider these additional identities

Even-Odd and Cofunction Identities

$$\sin(-A) = -\sin A \qquad \cos(-A) = \cos A \qquad \tan(-A) = -\tan A$$

$$\sin\left(\frac{\pi}{2}-A\right) = \cos A \qquad \cos\left(\frac{\pi}{2}-A\right) = \sin A \qquad \tan\left(\frac{\pi}{2}-A\right) = \cot A$$

$$\csc\left(\frac{\pi}{2}-A\right) = \sec A \qquad \sec\left(\frac{\pi}{2}-A\right) = \csc A \qquad \cot\left(\frac{\pi}{2}-A\right) = \tan A$$

We can use a combination of these identities to help solve some trigonometry problems that may look a little challenging. Let's see some examples.

Example 1: Find the exact value of $\cos 105°$

Solution 1: If we can achieve special angles, it simplifies things drastically

$$\cos 105° = \cos(60° + 45°) \quad \rightarrow \quad \cos 60°\cos 45° - \sin 60°\sin 45°$$

$$\cos 60°\cos 45° - \sin 60°\sin 45° = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

- May need to be able to work backwards too, these identities do not need to be memorized, but get fluid in their use

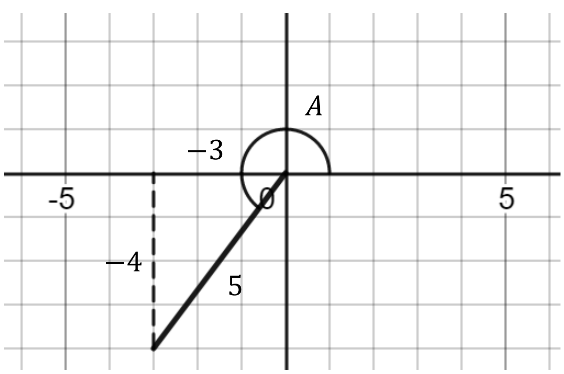
Example 2: Simplify $\dfrac{\tan\left(\frac{2\pi}{5}\right)-\tan\left(\frac{3\pi}{20}\right)}{1+\tan\left(\frac{2\pi}{5}\right)\tan\left(\frac{3\pi}{20}\right)}$

Solution 2: Working backwards we get…

$$\frac{\tan\left(\frac{2\pi}{5}\right)-\tan\left(\frac{3\pi}{20}\right)}{1+\tan\left(\frac{2\pi}{5}\right)\tan\left(\frac{3\pi}{20}\right)} = \tan\left(\frac{2\pi}{5}-\frac{3\pi}{20}\right)=\tan\left(\frac{\pi}{4}\right)=1$$

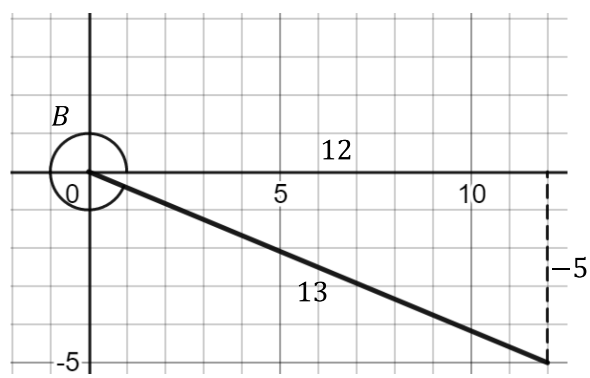
Example 3: Given $\sin A=-\frac{4}{5},\ A\ is\ in\ Q3.$ $\cos B=\frac{12}{13}, B$ in $Q4$. Find $\sin(A+B)$

Solution 3: Consider that we will need $\cos A\ \ and\ \sin B$



$a^2+b^2=r^2 \quad \rightarrow \quad a^2+(-4)^2=5^2$

$a^2=25-16=9 \quad \rightarrow \quad a=-3$



$a^2+b^2=r^2 \quad \rightarrow \quad 12^2+b^2=13^2$

$b^2=169-144=25 \quad \rightarrow \quad b=-5$

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$=\left(-\frac{4}{5}\right)\left(\frac{12}{13}\right)+\left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right)$$

$$=\left(-\frac{48}{65}\right)+\left(\frac{15}{65}\right)$$

$$=-\frac{25}{65} \quad \rightarrow \quad -\mathbf{\frac{5}{13}}$$

Example 4: Solve: $\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)=-1, \qquad 0 \le x < 2\pi$

Solution 4: Again, here we need to expand and simplify, then solve for x over the Domain provided.

$$\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)=-1$$

$$\left(\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}\right)+\left(\sin x\cos\frac{\pi}{4}-\cos x\sin\frac{\pi}{4}\right)=-1$$

$$2\sin x\cos\frac{\pi}{4}=-1$$

$$2\sin x\left(\frac{\sqrt{2}}{2}\right)=-1$$

$$2\sin x=-\frac{2}{\sqrt{2}} \quad \rightarrow \quad \sin x=-\frac{1}{\sqrt{2}}$$

This gives Special Angle of:

$$\frac{\pi}{4}$$

Sine is Negative in $Q3 \text{ and } Q4$.

Solutions are:

$$Q3: \frac{5\pi}{4} \qquad Q4: \frac{7\pi}{4}$$

Example 5: Prove the identity: $\sin(A+B)+\sin(A-B)=2\sin A\cos B$

Solution 5: Remember – Manipulate the more complicated side first and see where you end up

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$

$$(\sin A\cos B+\cos A\sin B)+(\sin A\cos B-\cos A\sin B)=$$

$$\sin A\cos B+\sin A\cos B+\cos A\sin B-\cos A\sin B=$$

$$2\sin A\cos B=$$

Example 6: Find the General Form of the Solution of: $2\tan x+\tan(\pi-3)=\sqrt{3}$

Solution 6: Work with the info we have

$$2\tan x+\tan(\pi-3)=\sqrt{3} \quad \rightarrow \quad 2\tan x+\frac{\tan\pi-\tan x}{1+\tan\pi\tan x}=\sqrt{3}$$

$$2\tan x+\frac{0-\tan x}{1+0(\tan x)}=\sqrt{3} \quad \rightarrow \quad 2\tan x-\frac{\tan x}{1}=\sqrt{3}$$

Tan Positive in $Q1$ and $Q3$, but also has a Period of π

$$\tan x=\sqrt{3} \quad \rightarrow \quad x=\tan^{-1}\sqrt{3}$$

$$x=\frac{\pi}{3} \text{ and } \frac{4\pi}{3} \quad (\text{differ by } \pi)$$

General Form:

$$\boldsymbol{x=\frac{\pi}{3}+\pi n, \quad n \text{ is an integer}}$$

- We can use Sum and Difference Identities to learn about Wave Function as well.

Example 7: Find the amplitude, period, and phase shift of:

$$f(x) = 3\sqrt{2}\sin 2x \cos\frac{\pi}{4} + 3\sqrt{2}\cos 2x \sin\frac{\pi}{4}$$

Solution 7: We need to write this as a single function. First look for Common Factors, then your Identities

$$f(x) = 3\sqrt{2}\sin 2x \cos\frac{\pi}{4} + 3\sqrt{2}\cos 2x \sin\frac{\pi}{4}$$

$$= 3\sqrt{2}\left(\sin 2x \cos\frac{\pi}{4} + \cos 2x \sin\frac{\pi}{4}\right)$$

$$= 3\sqrt{2}\sin\left(2x + \frac{\pi}{4}\right)$$

$$= 3\sqrt{2}\sin 2\left(x + \frac{\pi}{8}\right)$$

Amplitude: $3\sqrt{2}$ **Period:** $\frac{2\pi}{2} = \pi$ **Phase Shift:** $x = -\frac{\pi}{8}$

Example 8: Simplify: $\csc(90° - \theta)\sec(360° - \theta) - \tan(720° + \theta)\cot(450° - \theta)$

Solution 8: This question involves some of our Co-Function Identities and Sum/Difference Identities

$$\csc(90° - \theta)\sec(360° - \theta) = \tan(720° + \theta)\cot(450° - \theta)$$

Co-Function

$$\frac{1}{\sin(90° - \theta)} \cdot \frac{1}{\cos(360° - \theta)} = \tan(720° + \theta) \cdot \frac{\cos(450° - \theta)}{\sin(450° - \theta)}$$

$$\frac{1}{\cos\theta} \cdot \frac{1}{\cos 360° \cos\theta + \sin 360° \sin\theta} = \frac{\tan 720° + \tan\theta}{1 - \tan 720° \tan\theta} \cdot \frac{\cos 450° \cos\theta + \sin 450° \sin\theta}{\sin 450° \cos\theta - \cos 450° \sin\theta}$$

$\cos 360° = 1$

$\sin 360° = 0$

$\cos 450° = 0$

$\sin 450° = 1$

$\tan 720° = 0$

$$\frac{1}{\cos\theta} \cdot \frac{1}{\cos\theta + (0 \cdot \sin\theta)} = \frac{0 + \tan\theta}{1 - (0 \cdot \tan\theta)} \cdot \frac{(0 \cdot \cos\theta) + \sin\theta}{\cos\theta - (0 \cdot \sin\theta)}$$

$$\frac{1}{\cos\theta} \cdot \frac{1}{\cos\theta} - \tan\theta \cdot \frac{\sin\theta}{\cos\theta} \quad \rightarrow \quad \frac{1}{\cos^2\theta} - \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} \quad \rightarrow \quad \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

$$\frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} = \frac{1 - \sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\cos^2\theta} = 1$$

Section 7.4 – Practice Problems

1. Find the exact value of each expression.

a) $\sin 15°$

b) $\cos(-75°)$

c) $\tan\frac{5\pi}{12}$

d) $\cot\frac{11\pi}{12}$

e) $\sec\frac{19\pi}{12}$

f) $\csc(-105°)$

2. Simplify each expression

a) $\sin 24° \cos 36° + \cos 24° \sin 36°$

b) $\cos 55° \cos 10° + \sin 55° \sin 10°$

c) $\dfrac{\tan\frac{\pi}{5} - \tan\frac{\pi}{30}}{1 + \tan\frac{\pi}{5}\tan\frac{\pi}{30}}$

d) $\sin\frac{23\pi}{18}\cos\frac{\pi}{9} - \cos\frac{23\pi}{18}\sin\frac{\pi}{9}$

e) $\cos\frac{\pi}{8}\cos\frac{7\pi}{8} + \sin\frac{\pi}{8}\sin\frac{7\pi}{8}$

f) $\dfrac{\tan\frac{2\pi}{9} + \tan\frac{\pi}{9}}{1 - \tan\frac{2\pi}{9}\tan\frac{\pi}{9}}$

g) $\dfrac{\sin 3x}{\csc x} - \dfrac{\cos 3x}{\sec x}$

h) $\cos(A+B)\cos B + \sin(A+B)\sin B$

i) $\tan^2\left(\frac{\pi}{2} - x\right)\sec^2 x - \sin^2\left(\frac{\pi}{2} - x\right)\csc^2 x$

j) $\sin\left(\frac{\pi}{3} - x\right)\cos\left(\frac{\pi}{3} + x\right) + \cos\left(\frac{\pi}{3} - x\right)\sin\left(\frac{\pi}{3} + x\right)$

3. Find the exact value of each expression

a) $\tan x = 3,\ \ find \tan\left(x + \frac{\pi}{4}\right)$

b) $\sin x = \dfrac{4}{5}$, x is in $Q1$, find $\sin(x + \frac{\pi}{6})$

c) $\cos x = \frac{12}{13}$, x is in $Q1$, find $\cos(x + \frac{2\pi}{3})$

d) If both A and B are third quadrant angles, and $\cos B = -\frac{12}{13}$, what is the value of $\sin(A - B)$ if $\sin A = -\frac{3}{5}$

e) Given that $\sin A = \frac{12}{13}$ is in $Q2$, and $\sec B = \frac{5}{4}$, is in $Q4$, what is the value of $\cos(A + B)$

f) Given that $\tan A = \frac{5}{12}$ is in $Q3$, and $\cos B = -\frac{3}{5}$, is in $Q2$, what is the value of $\sin(A - B)$

4. Find the solution, $0° \leq \theta < 360°$, or $0 \leq x < 2\pi$, for each equation.

a) $\cos\theta\cos 10° - \sin\theta\sin 10° = \frac{1}{2}$

b) $\sin\theta\cos 12° + \cos\theta\cos 78° = \frac{\sqrt{3}}{2}$

c) $\cos 3x\cos x + \sin 3x\sin x = 0$

d) $2\tan x + \tan(\pi - x) = \sqrt{3}$

e) $\sqrt{2}\sin 3x\cos 2x = 1 + \sqrt{2}\cos 3x\sin 2x$

f) $\cos\left(x + \frac{\pi}{4}\right) + \cos\left(x - \frac{\pi}{4}\right) = 1$

5. Prove the identities

a) $\sin(A + B) - \sin(A - B) = 2\sin B\cos A$

b) $\dfrac{\sin(A + B)}{\cos(A - B)} = \dfrac{\cot A + \cot B}{1 + \cot A\cot B}$

c) $\frac{\sin(A-B)}{\sin B}+\frac{\cos(A-B)}{\cos B}=\frac{\sin A}{\sin B\cos B}$

d) $\frac{1+\tan A}{\tan\left(A+\frac{\pi}{4}\right)}=1-\tan A$

e) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$

f) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$

g) $\sec(A+B)=\frac{\sec A \sec B}{1-\tan A \tan B}$

h) $\csc(A-B)=\frac{\csc A \csc B}{\cot B-\cot A}$

6. Simplify the following statements

a) $\cos(90°-A)\sin(180°-B)+\cos(360°-A)\sin(90°-B)$

b) $\cos(A-90°)\sin(90°-B)-\sin(B-270°)\cos(90°-A)$

c) $\tan(90° - A)\tan(180° - A)\sec A + \csc B \sin(90° - B)\csc(90° - B)$

d) $\sec(180° - A)\csc(270° - A) - \cot(630° + A)\tan(540° - A)$

7. Find the Amplitude, Period, and Phase Shift of the following trigonometric functions.

a) $y = \cos 3x \cos x - \sin 3x \sin x$

b) $y = -2\sin 2x \cos\frac{\pi}{3} + 2\cos 2x \sin\frac{\pi}{3}$

c) $y = 3\sin\frac{\pi}{6}x\cos\frac{\pi}{3} + 3\cos\frac{\pi}{6}x\sin\frac{\pi}{3}$

d) $y = -\sin\frac{\pi}{4}x\sin\frac{\pi}{2} - \cos\frac{\pi}{4}x\cos\frac{\pi}{2}$

8. Consider the expression $\tan\left(\frac{\pi}{2}+x\right)$. Why can't we use the identity for $\tan(A+B)$ to express it as a function of x alone?

9. Demonstrate that $\tan\left(\frac{\pi}{2}+x\right)=-\cot x$

See Website for Detailed Answer Key

Extra Work Space