Section 7.3 – Trigonometric Equations

- Trigonometric Equations are different from Identities as they have a certain set of solutions
- Solving trigonometric equations is similar to solving algebraic equations
- We can solve for solutions of two types:
  - Conditional Solutions, where $0 \leq x < 2\pi$
  - General Form Solutions, where we consider all possible solutions

Example 1: Solve $2\sin x - 1 = 0$

a) Over $0 \leq x < 2\pi$ b) General Form

Solution 1: First we solve for x

Since Sine is Positive in both $Q1\ and\ Q2$, we have to consider the reference angle for:

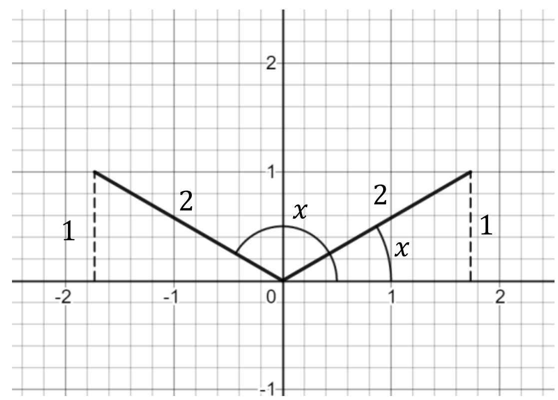
$$2\sin x - 1 = 0 \quad \rightarrow \quad \sin x = \frac{1}{2}$$

By Special Angles:

$$\sin^{-1}\left(\frac{1}{2}\right) = \left(\frac{\pi}{6}\right)$$

$$Q1: \; x = \frac{\pi}{6} \qquad Q2: \; x = \frac{5\pi}{6}$$

a) Solutions are: $\boldsymbol{\frac{\pi}{6}}$ **and** $\boldsymbol{\frac{5\pi}{6}}$



The General Form consider the solutions across the infinite length of the Sine Wave.

Since $\sin x$ has a Period of 2π. We have solutions every 2π. So we say $2\pi n$, where n is an integer.

b) $\boldsymbol{\frac{\pi}{6} + 2\pi n}$ **and** $\boldsymbol{\frac{5\pi}{6} + 2\pi n}$

Example 2: Solve $\cos x + \sqrt{2} = -\cos x$

a) Over $0 \leq x < 2\pi$ b) General Form

Solution 2: First we solve for x

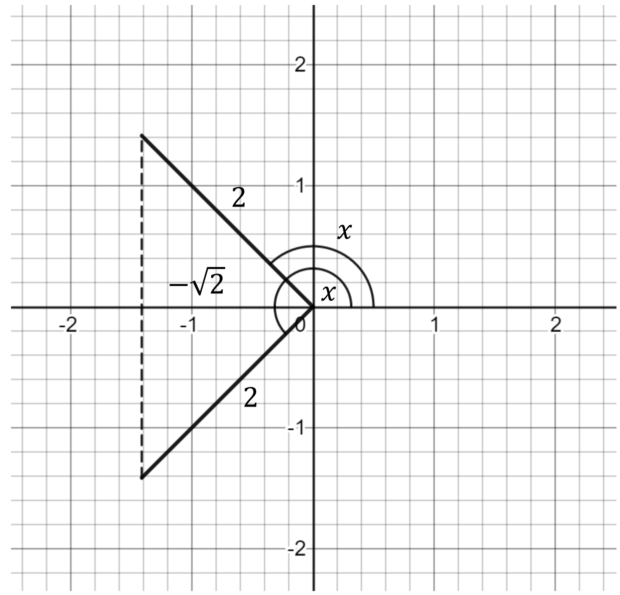
Since Cosine is Negative in both $Q2\ and\ Q3$, we have to consider the reference angle for:

$$\cos x + \sqrt{2} = -\cos x \quad \rightarrow \quad 2\cos x = -\sqrt{2} \quad \rightarrow \quad \cos x = -\frac{\sqrt{2}}{2}$$

By Special Angles:

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \left(\frac{\pi}{4}\right)$$

$$Q2: \; x = \frac{3\pi}{4} \qquad Q3: \; x = \frac{5\pi}{4}$$



a) Solutions are: $\boldsymbol{\frac{3\pi}{4}}$ **and** $\boldsymbol{\frac{5\pi}{4}}$

b) Since $\cos x$ has a Period of 2π. We have solutions every 2π. So we say $2\pi n$, where n is an integer.

$$\boldsymbol{\frac{3\pi}{4} + 2\pi n} \quad \textbf{and} \quad \boldsymbol{\frac{5\pi}{4} + 2\pi n}$$

Example 3: Solve $\tan x - \sqrt{3} = 0$

a) Over $0 \le x < 2\pi$ b) General Form

Solution 3: First solve for $\tan x$

$$\tan x - \sqrt{3} = 0 \quad \rightarrow \quad \tan x = \sqrt{3}$$

This occurs for Tangent in both $Q1$ and $Q3$. With a reference angle given by special angles of:

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

So, in $Q1$: $x = \frac{\pi}{3}$ in $Q3$: $x = \frac{4\pi}{3}$

a) **Solutions are:** $\boldsymbol{\frac{\pi}{3}}$ **and** $\boldsymbol{\frac{4\pi}{3}}$

b) **Tan has a Period of $\boldsymbol{\pi}$. So we get:**

$\boldsymbol{\frac{\pi}{3} + \pi n}$**, where $\boldsymbol{n}$ is an integer**

Example 4: Solve: $\sin x \tan x = 2\tan x$

b) Over $0 \le x < 2\pi$ b) General Form

Solution 4: First factor the equation

$\sin x \tan x = 2\tan x \quad \rightarrow \quad \sin x \tan x - 2\tan x = 0$

$\tan x\,(\sin x - 2) = 0$

Once factored, we need to consider when both factors are equal to 0.

$$\tan x = 0 \quad \rightarrow \quad x = \tan^{-1}(0) = 0$$

$$\sin x - 2 = 0 \quad \rightarrow \quad \sin x = 2 \quad \rightarrow \quad x = \sin^{-1}(2) = \emptyset$$

No Solution

So, for $\tan x$, with a period of π:

a) **Solutions are: $\mathbf{0, \pi}$**

b) **Tan has a Period of $\boldsymbol{\pi}$. So we get: $\boldsymbol{\pi n; n}$ is an integer**

Example 5: $\sec^2 x - \sec x - 2 = 0$

a) Over $0 \le x < 2\pi$ b) General Form

Solution 5: We have to factor this, just like we factor a quadratic.

$\sec^2 x - \sec x - 2 = 0 \quad \rightarrow \quad (\sec x - 2)(\sec x + 1) = 0$

So,

$$\sec x - 2 = 0$$

$$\sec x = 2 \qquad \textit{Pos, so Q1 and Q4}$$

$$x = \sec^{-1}(2)$$

$$x = \frac{\pi}{3}$$

And

$$x = \frac{5\pi}{3}$$

$$\sec x + 1 = 0$$

$$\sec x = -1 \qquad \textit{Quadrantal Angle}$$

$$x = \sec^{-1}(-1)$$

$$x = \pi$$

a) Solutions are: $\boldsymbol{\frac{\pi}{3}, \pi, \frac{5\pi}{3}}$

b) General Solution: $\boldsymbol{\frac{\pi}{3} + 2\pi n, \pi + 2\pi n, \frac{5\pi}{3} + 2\pi n}$

Notice: They differ by $\frac{2\pi}{3}$, so: $\boldsymbol{\frac{\pi}{3} + \frac{2\pi n}{3}}$ where n is an integer

Example 6: Solve: $2\cos^2 x + 3\sin x - 3 = 0$

a) Over $0 \le x < 2\pi$ b) General Form

Solution 6: We have to factor this, but we need the trig functions to be the same. Use identities.

$2\cos^2 x + 3\sin x - 3 = 0 \quad \rightarrow \quad 2(1 - \sin^2 x) + 3\sin x - 3 = 0$

$$2 - 2\sin^2 x + 3\sin x - 3 = 0$$

$$-2\sin^2 x + 3\sin x - 1 = 0$$

$$2\sin^2 x - 3\sin x + 1 = 0$$

$$(2\sin x - 1)(\sin x - 1) = 0$$

So,

$$2\sin x - 1 = 0$$

$\sin x = \frac{1}{2}$ $\quad$ *Pos, so Q1 and Q2*

$$x = \sin^{-1}\left(\frac{1}{2}\right)$$

$$x = \frac{\pi}{6}$$

And

$$x = \frac{5\pi}{6}$$

$$\sin x - 1 = 0$$

$\sin x = 1$ $\quad$ *Quadrantal Angle*

$$x = \sin^{-1}(1)$$

$$x = \frac{3\pi}{2}$$

$$x = \frac{9\pi}{6}$$

Just getting a Common Denominator

a) Solutions are: $\boldsymbol{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}}$

b) Notice they differ by $\frac{4\pi}{6} = \frac{2\pi}{3}$, so: $\boldsymbol{\frac{\pi}{6} + \frac{2\pi n}{3}}$

where n is an integer.

Example 7: Solve: $\cos x + 1 = \sin x$

a) Over $0 \le x < 2\pi$ b) General Form

Solution 7: We have to factor this, but we need the trig functions to be the same. Use identities. But for this one, we will have to square both sides first. Look out, because this creates potential extraneous solutions.

$\cos x + 1 = \sin x \quad \rightarrow \quad (\cos x + 1)^2 = \sin^2 x$

$$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x$$

$$2\cos^2 x + 2\cos x = 0$$

$$2\cos x\,(\cos x + 1) = 0$$

$$\cos x\,(\cos x + 1) = 0$$

So,

$$\cos x = 0$$

$\cos x = 0$ $\quad$ *Quadrantal Angles*

$$x = \cos^{-1}(0)$$

$$x = \frac{\pi}{2}$$

And

$$x = \frac{3\pi}{2}$$

$$\cos x + 1 = 0$$

$\cos x = -1$ $\quad$ *Quadrantal Angle*

$$x = \cos^{-1}(-1)$$

$$x = \pi$$

Need to Check for Extraneous Solutions

$\cos x + 1 = \sin x$	$\cos x + 1 = \sin x$	$\cos x + 1 = \sin x$
$\cos\frac{\pi}{2} + 1 = \sin\frac{\pi}{2}$	$\cos\frac{3\pi}{2} + 1 = \sin\frac{3\pi}{2}$	$\cos\pi + 1 = \sin\pi$
$0 + 1 = 1$	$0 + 1 = -1$	$-1 + 1 = 0$
$1 = 1$	$1 = -1$	$0 = 0$
Accept	*Reject*	*Accept*

a) Solutions are: $\boldsymbol{\frac{\pi}{2}, \pi}$

b) General Solution: $\boldsymbol{\frac{\pi}{2} + 2\pi n, \pi + 2\pi n}$

Example 8: Solve: $2\sin 3\theta + 1 = 0$

a) Over $0 \leq x < 360°$ b) General Form

Solution 8: We have to factor this, but look out. We do not have θ, we have 3θ. See what happens.

$$2\sin 3\theta + 1 = 0 \quad \rightarrow \quad 2\sin 3\theta = -1 \quad \rightarrow \quad \sin 3\theta = -\frac{1}{2}$$

So,

$$\sin 3\theta = -\frac{1}{2}$$

$$3\theta = \sin^{-1}\left(-\frac{1}{2}\right)$$

Negative in $Q3$ and $Q4$

Reference Angle of 30°

$$3\theta = 210°$$

And

$$3\theta = 330°$$

a) Solutions are: $3\theta = 210°$ and $3\theta = 330°$ but we want θ and have to consider $0° \leq \theta < 360°$

So,

$3\theta = 210°$ Gives $\boldsymbol{\theta = 70°}$	$3\theta = 210° + 360°$ Gives $\theta = 70° + 120° = \mathbf{190°}$	$3\theta = 210° + (2)360°$ Gives $\theta = 70° + (2)120° = \mathbf{310°}$
$3\theta = 330°$ Gives $\boldsymbol{\theta = 110°}$	$3\theta = 330° + 360°$ Gives $\theta = 110° + 120° = \mathbf{230°}$	$3\theta = 330° + (2)360°$ Gives $\theta = 110° + (2)120° = \mathbf{350°}$

a) Solutions are: $\mathbf{70°, 110°, 190°, 230°, 310°, 350°}$

b) General Solutions are:

$3\theta = 210° + 360°n \rightarrow \boldsymbol{\theta = 70° + 120°n}$ and $3\theta = 330° + 360°n \rightarrow \boldsymbol{\theta = 110° + 120°n}$

Example 9: Solve: $\cos^2 x - 3\cos x - 2 = 0$

a) Over $0 \leq x < 360°$ b) General Form

Solution 9: This is not easily factorable so we use the Quadratic Equation. $a = 1$, $b = -3$, $c = -2$

$$\cos x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cos x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2(1)}$$

$$\cos x = \frac{3 \pm \sqrt{9 + 8}}{2}$$

$$= 3.5616 \quad or \quad -0.5616$$

Since $\cos x$ exists only between -1 and 1, we reject 3.5616

We need the reference angle for $\cos x = -0.5616$, we use the positive value.

$$x = \cos^{-1}(0.5616) = 0.9745$$

Consider Cosine is Negative is $Q2$ and $Q3$.

For $Q2$: $x = \pi - 0.9745 = 2.1671$

For $Q3$: $x = \pi + 0.9745 = 4.1161$

Therefore:

a) Solutions are: $\mathbf{2.1671} \quad \boldsymbol{and} \quad \mathbf{4.1161}$

b) General Solutions are:

$$\mathbf{2.1671 + 2\pi n} \quad \textbf{and} \quad \mathbf{4.1161 + 2\pi n}$$

Example 10: Solve: $6\sin^2 2x - \sin 2x - 1 = 0$

a) Over $0 \le x < 360°$ b) General Form

Solution 10: We have to factor this, but we do not have x, we have $2x$. See what happens.

$6\sin^2 2x - \sin 2x - 1 = 0 \quad \rightarrow \quad 6\sin^2 2x - 3\sin 2x + 2\sin 2x - 1 = 0$

$$(6\sin^2 2x - 3\sin 2x)\,(+2\sin 2x - 1) = 0$$

Factor by Grouping

$$3\sin 2x\,(2\sin 2x - 1) + 1(2\sin 2x - 1) = 0$$

$$(3\sin 2x + 1)(2\sin x - 1) = 0$$

So,

Positive in $Q1$ and $Q2$ $\quad \sin 2x = \frac{1}{2} \quad$ *and* $\quad \sin 2x = -\frac{1}{3}$ $\quad$ *Negative* in $Q3$ and $Q4$

For reference angles:

Use positive values of the Ratios

$2x = \sin^{-1}\left(\frac{1}{2}\right) \quad \rightarrow \quad 2x = \frac{\pi}{6}$

$Q1$: $2x = \frac{\pi}{6}$ and $Q2$: $2x = \frac{5\pi}{6}$ $\quad$ Both occur in cycles of 2π

$$2x = \sin^{-1}\left(\frac{1}{3}\right) \quad \rightarrow \quad 2x = 0.3398$$

$Q3$: $2x = \pi + 0.3398 = 3.481$ and $Q4$: $2x = 2\pi - 0.3398 = 5.943$

Both occur in cycles of 2π

a) Solutions are: $2x = \frac{\pi}{6}$ and $2x = 0.3398$ but we want x and have to consider $0 \leq x < 2\pi$

$2x = \frac{\pi}{6}$ Gives $\boldsymbol{x = \frac{\pi}{12}}$	$2x = \frac{\pi}{6} + 2\pi$ Gives $\boldsymbol{\theta = \frac{\pi}{12} + \pi = \frac{13\pi}{12}}$	$2x = \frac{5\pi}{6}$ Gives $\boldsymbol{x = \frac{5\pi}{12}}$	$2x = \frac{5\pi}{6} + 2\pi$ Gives $\boldsymbol{\theta = \frac{5\pi}{12} + \pi = \frac{17\pi}{12}}$
$2x = 3.481$ Gives $\boldsymbol{x = 1.741}$	$2x = 3.481 + 2\pi$ Gives $\boldsymbol{x = 1.741 + \pi = 4.883}$	$2x = 5.943$ Gives $\boldsymbol{x = 2.972}$	$2x = 5.943 + 2\pi$ Gives $\boldsymbol{x = 2.972 + \pi = 6.113}$

a) Solutions are: $\boldsymbol{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, 1.741, 2.972, 4.883, 6.113}$

b) General Solutions are:

$2x = \frac{\pi}{6} + 2n\pi$ $x = \frac{\pi}{12} + n\pi$	$2x = \frac{5\pi}{6} + 2n\pi$ $x = \frac{5\pi}{12} + n\pi$	$2x = 3.481 + 2\pi n$ $x = 1.741 + \pi n$	$2x = 5.943 + 2\pi n$ $x = 2.972 + \pi n$

Therefore, General Solutions are:

$$\frac{\pi}{12} + n\pi$$

$$\frac{5\pi}{12} + n\pi$$

$$1.741 + \pi n$$

$$2.972 + \pi n$$

Section 7.3 – Practice Problems

1. Solve for: i) All solutions over $0 \leq x < 2\pi$ ii) The General Form

a) $\sin x = \frac{\sqrt{3}}{2}$

b) $\cos x = \frac{\sqrt{2}}{2}$

c) $\tan x = \frac{1}{\sqrt{3}}$

d) $\cot x = \frac{1}{\sqrt{3}}$

e) $\sec x = \frac{2}{\sqrt{3}}$

f) $\csc x = 2$

g) $\sin x = -\frac{1}{2}$

h) $\cos x = -1$

i) $\tan x = -\sqrt{3}$

j) $\cot x = 0$

k) $\sec x = -\sqrt{2}$

l) $\csc x = -\dfrac{2}{\sqrt{3}}$

2. Solve for: i) All solutions over $0 \leq x < 2\pi$ ii) The General Form

a) $\sin x = 0.6234$

b) $\cos x = 0.4821$

c) $\tan x = 1.7258$

d) $\cot x = 0.7238$

e) $\sec x = 3.1743$

f) $\csc x = 1.5243$

g) $\sin x = -0.4173$

h) $\cos x = -0.4821$

i) $\tan x = -0.3124$

j) $\cot x = -1.1482$

k) $\sec x = -1.9105$

l) $\csc x = -\ 2.3124$

3. How many solutions do the following equations have for $0 \leq x < 2\pi$?

a) $\sin 3x = -\frac{1}{4}$

b) $\sin 3x = -1$

c) $\sin \frac{1}{2}x = \frac{1}{3}$

d) $\cos \frac{1}{2}x = \frac{1}{3}$

e) $\tan^2 2x = 1$

f) $\sin bx = \dfrac{1}{2}$

4. Solve for: i) All solutions over $0 \le x < 2\pi$ ii) The General Form

a) $\sin 2x = \dfrac{\sqrt{3}}{2}$

b) $\tan 3x = -1$

c) $\sec\frac{x}{2} = -\frac{2}{\sqrt{3}}$

d) $\sin 2x = -0.4173$

e) $\tan 2x = 1.7258$

f) $\tan bx = 1.7258,\ b\ is\ an\ integer$

5. Solve the given equations algebraically; provide exact answers when possible

i) All solutions over $0 \leq x < 2\pi$ ii) The General Form

a) $2\cos x + 1 = 0$

b) $(2\sin x - 1)(\cos x + 1) = 0$

c) $\sqrt{2}\cos^2 x - \cos x = 0$

d) $4\sin^2 x = 3$

e) $\sin^2 x = \sin x$

f) $6\sin^2 x + 11\sin x - 10 = 0$

g) $5\cos^2 x + 6\cos x - 8 = 0$

h) $2\cos^2 x - \cos x = 1$

i) $2\cos^2 x - 3\cos x - 2 = 0$

j) $2\tan^2 x + 5\tan x + 2 = 0$

k) $\tan^2 x - 2\tan x - 3 = 0$

l) $\cot^2 x - \cot x - 6 = 0$

m) $\tan x - 2\tan x \cdot \sin x = 0$

n) $3\sin^2 x + 4\sin x - 4 = 0$

o) $\sec^2 x - 3\sec x + 2 = 0$

p) $2\cos^2 x - 3\sin x - 3 = 0$

q) $3\csc x - \sin x - 2 = 0$

r) $3\sin x = \sqrt{3}\cos x$

s) $\sin x \tan 2x = \sin x$

t) $3\sin^2 2x - 2\sin 2x - 1 = 0$

6. Solve using Desmos.

a) $\tan x - \sin 3x = 1, \qquad 0 \leq x < 2\pi$

b) $\sin 3x - \cos 2x = -1\,, \; 0 \leq x < 2\pi$

c) $\cot 2x + \tan \frac{1}{2} x = 0, \quad 0 \leq x < 2\pi$

See Website for Detailed Answer Key

Extra Work Space