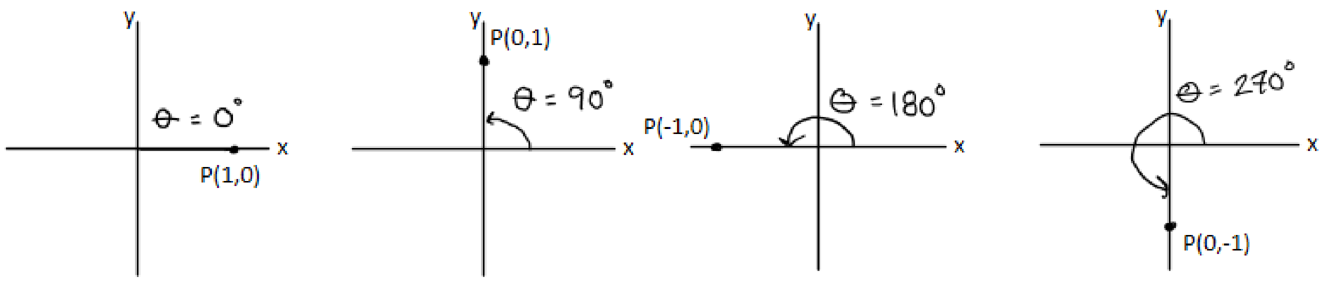
Section 7.3 – Special Angles

Quadrantal Angles

- Quadrantal angle are the easiest to calculate
- They are the angles where the **terminal arm is on** a the $\boldsymbol{x-axis\ or\ y-axis}$. $0° \le \theta \le 360°$



- The easiest points to choose are $\mathbf{1}\ \boldsymbol{unit}$ from the origin. **Where** $\boldsymbol{r} = \sqrt{\boldsymbol{x^2+y^2}} = \mathbf{1}$ **a positive number**

Example 1: Evaluate each trigonometric function.

a) $\tan 0°$ b) $\cos 180°$ c) $\sin 270°$ d) $\tan 90°$

Solution 1:

a) Any point may be selected on the terminal side of $\theta = 0°$. But simplify your life and choose $P(1,0)$.

$$r = \sqrt{x^2+y^2} = \sqrt{1^2+0^2} = 1$$

$$\boldsymbol{\tan 0° = \frac{y}{x} = \frac{0}{1} = 0}$$

b) Any point may be selected on the terminal side of $\theta = 180°$. But simplify your life and choose $P(-1,0)$.

$$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+0^2} = 1$$

$$\boldsymbol{\cos 180° = \frac{x}{r} = \frac{-1}{1} = -1}$$

c) Any point may be selected on the terminal side of $\theta = 270°$. But simplify your life and choose $P(0,-1)$.

$$r = \sqrt{x^2+y^2} = \sqrt{0^2+(-1)^2} = 1$$

$$\boldsymbol{\sin 270° = \frac{y}{r} = \frac{-1}{1} = -1}$$

d) Any point may be selected on the terminal side of $\theta = 90°$. But simplify your life and choose $P(0,1)$.

$$r = \sqrt{x^2+y^2} = \sqrt{0^2+(1)^2} = 1$$

$$\boldsymbol{\tan 90° = \frac{y}{x} = \frac{1}{0} = Undefined}$$

Special Angles: $30°, 45°, and\ 60°$

- Two triangles in trigonometry are especially significant, we can calculate them exactly
- They are the $45° - 45° - 90°$ triangle and the $30° - 60° - 90°$ triangle.

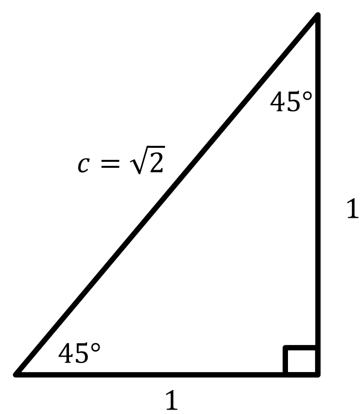
The $45° - 45° - 90°$ Triangle

- Since the triangle has two equal angles, it is an isosceles
- Since trigonometric function are based on ratios we can use any numbers, use 1 for simplicity
- By Pythagoras' Theorem:

$$c^2 = a^2 + b^2 = 1^2 + 1^2 \qquad \rightarrow \qquad c = \sqrt{2}$$

Therefore,

$$Sin\ 45° = \frac{1}{\sqrt{2}} \qquad Cos\ 45° = \frac{1}{\sqrt{2}} \qquad Tan\ 45° = \frac{1}{1} = 1$$



The $30° - 60° - 90°$ Triangle

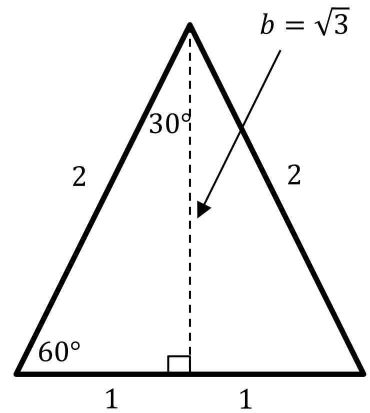
- Consider an equilateral triangle with all sides equal to 2
- Draw an altitude from a base to split the opposite 60 in half
- By Pythagoras' Theorem:

$$c^2 - a^2 = b^2 \qquad \rightarrow \qquad 2^2 - 1^2 = b^2 \qquad \rightarrow \qquad b = \sqrt{3}$$

Therefore,

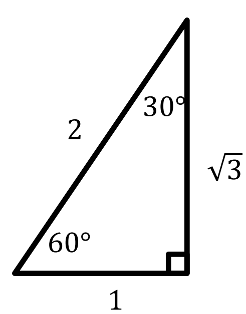
$$Sin\ 30° = \frac{1}{2} \qquad Cos\ 30° = \frac{\sqrt{3}}{2} \qquad Tan\ 30° = \frac{1}{\sqrt{3}}$$

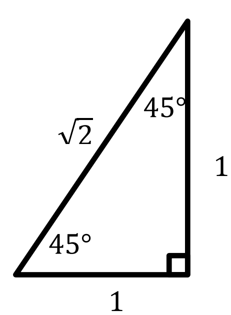
$$Sin\ 60° = \frac{\sqrt{3}}{2} \qquad Cos\ 60° = \frac{1}{2} \qquad Tan\ 60° = \sqrt{3}$$



Summary of Special Angles

θ	$Sin\ \theta$	$Cos\ \theta$	$Tan\ \theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

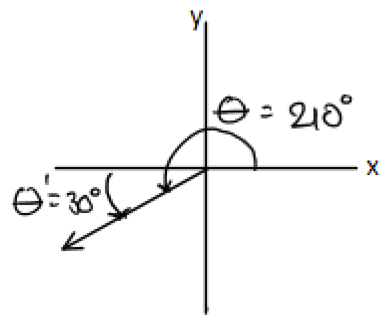




Example 2: Evaluate $\sin 210°$

Solution 2: We need to consider the reference angle. The reference angle of $210°$ is $30°$

We have a special angle as a reference angle



From special angles: $\sin 30° = \frac{1}{2}$

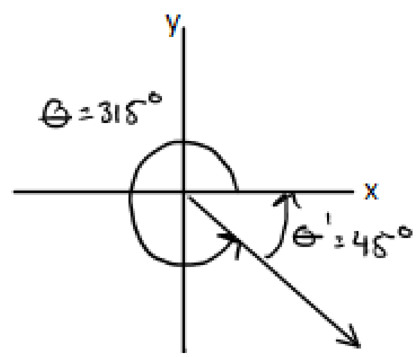
But since $210°$ is in $Q3$, sine is negative.

So… $\sin 210° = -\sin 30° = -\frac{1}{2}$

Example 3: Evaluate $\cos 315°$

Solution 3: We need to consider the reference angle. The reference angle of $315°$ is $45°$

We have a special angle as a reference angle



From special angles: $\cos 45° = \frac{1}{\sqrt{2}}$

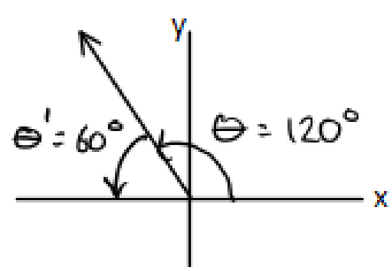
But since $315°$ is in $Q4$, cosine is positive.

So… $\cos 315° = \cos 45° = \frac{1}{\sqrt{2}}$

Example 4: Evaluate $\tan 120°$

Solution 4: We need to consider the reference angle. The reference angle of $120°$ is $60°$

We have a special angle as a reference angle

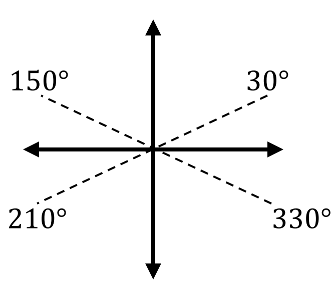


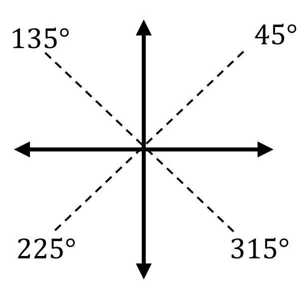
From special angles: $\tan 60° = \sqrt{3}$

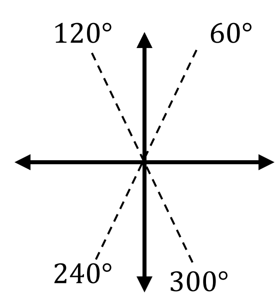
But since $120°$ is in $Q2$, tangent is negative.

So… $\tan 120° = -\tan 60° = -\sqrt{3}$

- There are specific values in each quadrant to reflect the special angle values $30°, 45°, and\ 60°$
- All that changes in the sign of the trigonometric function in the given quadrant







Example 5: Evaluate $\sin 30°$, $\sin 150°$, $\sin 210°$, $\sin 330°$

Solution 5: The **reference angle** for $30°, 150°, 210°, and\ 330°$ $\boldsymbol{is\ 30°}$

Sine is **positive in $Q1\ and\ Q2$**, and **negative in $Q3\ and\ Q4$**

Therefore: $\sin 30° = \frac{1}{2}$, $\sin 150° = \frac{1}{2}$, $\sin 210° = -\frac{1}{2}$, $\sin 330° = -\frac{1}{2}$

Example 6: Evaluate $\cos 45°$, $\cos 135°$, $\cos 225°$, $\cos 315°$

Solution 6: The **reference angle** for $45°, 135°, 225°, and\ 315°$ $\boldsymbol{is\ 45°}$

Cosine **is positive in $Q1\ and\ Q4$**, and **negative in $Q2\ and\ Q3$**

Therefore: $\cos 45° = \frac{1}{\sqrt{2}}$, $\cos 135° = -\frac{1}{\sqrt{2}}$, $\cos 225° = -\frac{1}{\sqrt{2}}$, $\cos 315° = \frac{1}{\sqrt{2}}$

Example 7: Evaluate $\tan 60°$, $\tan 120°$, $\tan 240°$, $\tan 300°$

Solution 7: The **reference angle** for $60°, 120°, 240°, and\ 300°$ $\boldsymbol{is\ 60°}$

Tangent is **positive in $Q1\ and\ Q3$**, and **negative in $Q2\ and\ Q4$**

Therefore: $\tan 60° = \sqrt{3}$, $\tan 120° = -\sqrt{3}$, $\tan 240° = \sqrt{3}$, $\tan 300° = -\sqrt{3}$

Summary of Special Angles

Here is a Chart of Special Angles Values for the $30° - 60° - 90°\ and\ 45° - 45° - 90°$ triangles. The goal is not to memorize this table, but to become fluent in the use and measurements during your work.

	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞ $Undefined$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$

	$180°$	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$
$\sin\theta$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$
$\cos\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞ $Undefined$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$

- The next examples are the **reverse process**. Given a ratio, there are **multiple correct results**.
- You need to **consider reference angles** and **Quadrant Location** (Sign of the trigonometric ratio)

Example 8: Find all the angles such that $\sin\theta = \frac{\sqrt{3}}{2} \quad 0° \leq \theta < 360°$

Solution 8: For $\frac{\sqrt{3}}{2}$ Sine has a reference angle of 60°

Sine **is positive in $Q1$ and $Q2$**, so $\boldsymbol{\theta = 60°}$ *and* $\boldsymbol{\theta = 180° - 60° = 120°}$

Therefore: $\sin 60° = \frac{\sqrt{3}}{2}$ *and* $\sin 120° = \frac{\sqrt{3}}{2}$

Example 9: Find all the angles such that $\cos\theta = -\frac{1}{\sqrt{2}} \quad 0° \leq \theta < 360°$

Solution 9: For $\frac{1}{\sqrt{2}}$ Cosine has a reference angle of 45°

Cosine **is negative in $Q2$ and $Q3$**, so $\boldsymbol{\theta = 180 - 45° = 135°}$ *and* $\boldsymbol{\theta = 180° + 45° = 225°}$

Therefore: $\cos 135° = -\frac{1}{\sqrt{2}}$ *and* $\cos 225° = -\frac{1}{\sqrt{2}}$

Example 10: Find the smallest positive angle such that $\tan\theta = -\frac{1}{\sqrt{3}} \quad 0° \leq \theta < 360°$

Solution 10: For $\frac{1}{\sqrt{3}}$ Tangent has a reference angle of 30°

Tangent **is negative in $Q2$ and $Q4$**, so the smallest angle is found in $\boldsymbol{Q2}$

$\boldsymbol{\theta = 180 - 30° = 150°}$

Example 11: Find all the angles such that $\cos\theta = 0.632, \quad 0° \leq \theta < 360°$

Solution 11: For $0.632 \quad \theta = cos^{-1}(0.632) = 50.8°$ therefore the reference angle of 50.8°

Cosine **is positive in $Q1$ and $Q4$**, so $\boldsymbol{\theta = 50.8°}$ *and* $\boldsymbol{\theta = 360° - 50.8° = 309.2°}$

Therefore: $\cos 50.8° = 0.632$ *and* $\cos 309.2° = 0.632$

Example 12: Find the smallest positive angle such that $\sin\theta = -0.711 \qquad 0° \leq \theta < 360°$

Solution 12: For $0.711 \quad \theta = sin^{-1}(0.711) = 45.3°$ therefore the reference angle of $45.3°$

*Do not use $sin^{-1}(0.711) \;\; \rightarrow \quad \textit{the negative is for the quadrant location}$

Sine **is negative in $\boldsymbol{Q3}$ and $\boldsymbol{Q4}$**, so the smallest angle is found in $\boldsymbol{Q3}$

$\boldsymbol{\theta = 180 + 45.3° = 225.3°}$

Example 13: Find the exact area of the given triangle. Side length are $9cm \text{ and } 11cm$ with an angle of 60° between the two sides.

Solution 13: A drawing helps visualize.

Recall that the Area of a Triangle is given by:

$$A = \frac{1}{2}bh$$

We know $b = 11$, but need to calculate h

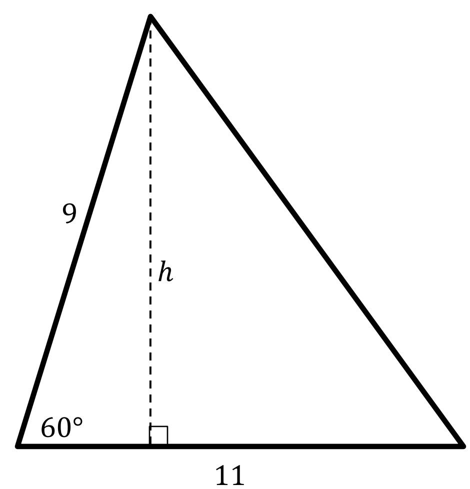
$$\sin 60° = \frac{h}{9} \rightarrow 9\sin 60° = h$$

$$h = 9\left(\frac{\sqrt{3}}{2}\right) = \frac{9\sqrt{3}}{2}$$

so…

$$A = \frac{1}{2}(11)(9)(\frac{\sqrt{3}}{2})$$

$$\boldsymbol{A = \frac{99\sqrt{3}}{4}cm^2}$$



Section 7.3 – Practice Problems

Find the reference angle for each given angle.

1. $300°$	2. $135°$
3. $240°$	4. $120°$
5. $330°$	6. $150°$
7. $111°$	8. $200°$
9. $280°$	10. $180°$
11. $73°$	12. $91°$
13. $179°$	14. $270°$

Find the angle θ, for each reference angle in the desired Quadrant

15. $30°, Q2$	16. $45°, Q3$
17. $60°, Q4$	18. $30°, Q3$
19. $45°, Q2$	20. $60°, Q2$

21. $30°, Q4$	22. $45°, Q4$
23. $60°, Q3$	24. $37°, Q2$
25. $37°, Q3$	26. $37°, Q4$

Find all θ, $0° \leq \theta < 360°$, which satisfy each equation

27. $\sin\theta = \frac{\sqrt{3}}{2}$	28. $\cos\theta = \frac{\sqrt{3}}{2}$
29. $\tan\theta = -\frac{1}{\sqrt{3}}$	30. $\sin\theta = -\frac{1}{\sqrt{2}}$
31. $\cos\theta = -\frac{1}{\sqrt{2}}$	32. $\tan\theta = -1$
33. $\sin\theta = 0$	34. $\cos\theta = 0$

35. $\tan\theta = 0$	36. $\sin\theta = -1$
37. $\cos\theta = -\frac{1}{2}$	38. $\tan\theta = \sqrt{3}$

Find to one decimal place, all θ, $0° \le \theta < 360°$, which satisfy each equation

39. $\sin\theta = 0.253$	40. $\cos\theta = 0.425$
41. $\tan\theta = 2$	42. $\sin\theta = -0.625$
43. $\cos\theta = -0.738$	44. $\tan\theta = -0.543$

Find the smallest positive angle θ, $0° \le \theta < 360°$, which satisfy each equation

45. $\sin\theta = -\frac{1}{2}$	46. $\cos\theta = -\frac{1}{2}$

47. $\tan\theta = -1$	48. $\sin\theta = -\frac{1}{\sqrt{2}}$
49. $\cos\theta = -\frac{1}{\sqrt{2}}$	50. $\tan\theta = -\sqrt{3}$
51. $\sin\theta = -\frac{\sqrt{3}}{2}$	52. $\cos\theta = -\frac{\sqrt{3}}{2}$

53. Find the area of a triangle with sides of length $5cm\ and\ 10cm$, and an angle of $110°$ between them.

54. A triangle has an area of $15mm^2$, and two sides of the triangle are $6mm\ and\ 8mm.$ Find the acute angle between the two sides of the triangle.

55. Find the area of an equilateral triangle (all sides and angles the same) with sides of $10cm$ in length. (Give an exact answer, no decimals)

Answer Key – Section 7.3

1. 60°	39. 14.7°, 165.3°
2. 45°	40. 64.8°, 295.2°
3. 60°	41. 63.4°, 243.4°
4. 60°	42. 218.7°, 321.3°
5. 30°	43. 137.6°, 222.4°
6. 30°	44. 151.5°, 331.5°
7. 69°	45. 210°
8. 20°	46. 120°
9. 80°	47. 135°
10. 0°	48. 225°
11. 73°	49. 135°
12. 89°	50. 120°
13. 1°	51. 240°
14. 90°	52. 150°
15. 150°	53. $23.49cm^2$
16. 225°	54. 38.7°
17. 300°	55. $25\sqrt{3}cm^2$
18. 210°	
19. 135°	
20. 120°	
21. 330°	
22. 315°	
23. 240°	
24. 143°	
25. 217°	
26. 323°	
27. 60°, 120°	
28. 30°, 330°	
29. 150°, 330°	
30. 225°, 315°	
31. 135°, 225°	
32. 135°, 315°	
33. 0°, 180°	
34. 90°, 270°	
35. 0°, 180°	
36. 270°	
37. 120°, 240°	
38. 60°, 240°	

Extra Work Space