Section 7.3 – Practice Problems

1. Find the derivative of the following functions

a) $y = 3\tan 2x$

$y' = 3\sec^2 2x \cdot 2$

$= \boxed{6\sec^2 2x}$

b) $y = \frac{1}{3}\cot 9x$

$y' = \frac{1}{3}(-\csc^2 9x)\cdot 9$

$= \boxed{-3\csc^2 9x}$

c) $y = 12\sec\frac{1}{4}x$

$12\sec\frac{1}{4}x\tan\frac{1}{4}x \cdot \frac{1}{4}$

$\boxed{3\sec\frac{1}{4}x\tan\frac{1}{4}x}$

d) $y = -\frac{1}{4}\csc(-8x)$

$-\frac{1}{4}(-\csc(-8x)\cot(-8x)) \cdot -8$

$\boxed{-2\csc(-8x)\cot(-8x)}$

e) $y = \tan x^2$

$y' = \sec^2 x^2 \cdot 2x$

$\Rightarrow \boxed{2x\sec^2 x^2}$

f) $y = \tan^2 x$

$y = (\tan x)^2$

$\boxed{y' = 2\tan x \cdot \sec^2 x}$

g) $y = \sec \sqrt[3]{x}$

$y = \sec x^{\frac{1}{3}}$

$y' = \sec x^{\frac{1}{3}} \tan x^{\frac{1}{3}} \cdot \frac{1}{3}x^{-2/3}$

$= \boxed{\dfrac{\sec\sqrt[3]{x}\tan\sqrt[3]{x}}{3x^{2/3}}}$

h) $y = x^2 \csc x$

$y' = x^2(-\csc x \cot x) + 2x\csc x$

$= -x^2\csc x\cot x + 2x\csc x$

$\boxed{x\csc x(2 - x\cot x)}$

i) $y = \cot^3(1-2x)^2$

$y = \left(\cot(1-2x)^2\right)^3$

$y' = 3\left(\cot(1-2x)^2\right)^2 \cdot \left(-\csc^2(1-2x)^2\right) \cdot 2(1-2x) \cdot (-2)$

$= \boxed{12(1-2x)\csc^2(1-2x)^2\cot^2(1-2x)^2}$

j) $y = \sec^2 x - \tan^2 x$

$y' = 2\sec x \cdot \sec x\tan x - 2\tan x\sec^2 x$

$y' = 2\sec^2 x\tan x - 2\sec^2 x\tan x$

$\boxed{y' = 0}$

k) $y = \dfrac{1}{\sqrt{(\sec 2x - 1)^3}}$

$y = (\sec 2x - 1)^{-3/2}$

$y' = \dfrac{-3}{2(\sec 2x-1)^{5/2}} \cdot \sec 2x\tan 2x \cdot 2$

$= \boxed{\dfrac{-3\sec 2x\tan 2x}{(\sec 2x-1)^{5/2}}}$

l) $y = \dfrac{x^2\tan x}{\sec x} \rightarrow \dfrac{x^2\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} \rightarrow \dfrac{x^2\sin x}{\cos x} \cdot \cos x$

$= x^2\sin x$

$y' = x^2\cos x + 2x\sin x$

$= \boxed{x(x\cos x + 2\sin x)}$

m) $y = 2x(\sqrt{x} - \cot x)$

$y' = 2x\left(\frac{1}{2\sqrt{x}} + \csc^2 x\right) + 2(\sqrt{x} - \cot x)$

$= \sqrt{x} + 2x\csc^2 x + 2\sqrt{x} - 2\cot x$

$= \boxed{3\sqrt{x} + 2x\csc^2 x - 2\cot x}$

n) $y = \sin(\tan x)$

$y' = \cos(\tan x) \cdot \sec^2 x$

$= \boxed{\sec^2 x \cos(\tan x)}$

o) $y = \tan^2(\cos x)$

$y' = 2\tan(\cos x) \cdot \sec^2(\cos x) \cdot -\sin x$

$= \boxed{-2\sin x \tan(\cos x)\sec^2(\cos x)}$

p) $y = [\tan(x^2 - x)^{-2}]^{-3}$

$y' = -3[\tan(x^2-x)^{-2}]^{-4} \cdot \sec^2(x^2-x)^{-2} \cdot -2(x^2-x)^{-3} \cdot (2x-1)$

$\boxed{y' = \frac{6(2x-1)\sec^2(x^2-x)^{-2}}{(x^2-x)^3\tan^4}}$

2. Find $\frac{dy}{dx}$

a) $\tan x + \sec y - y = 0$

$\sec^2 x + \sec y \tan y \frac{dy}{dx} - \frac{dy}{dx} = 0$

$\frac{dy}{dx}\sec y \tan y - \frac{dy}{dx} = -\sec^2 x$

$\frac{dy}{dx}(\sec y \tan y - 1) = -\sec^2 x$

$\boxed{\frac{dy}{dx} = -\frac{\sec^2 x}{\sec y \tan y - 1}}$

b) $\tan 2x = \cos 3y$

$$\sec^2 2x \cdot 2 = -\sin 3y \cdot 3\frac{dy}{dx}$$

$$2\sec^2 2x = -3\sin 3y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \boxed{-\frac{2\sec^2 2x}{3\sin 3y}}$$

c) $\cot(x+y) + \cot x + \cot y = 0$

$$-\csc^2(x+y)\cdot\left(1+\frac{dy}{dx}\right) + -\csc^2 x + -\csc^2 y\frac{dy}{dx} = 0$$

$$-\csc^2(x+y) + \frac{dy}{dx}(-\csc^2(x+y)) - \csc^2 x - \csc^2 y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(-\csc^2(x+y)) - \csc^2 y\frac{dy}{dx} = \csc^2 x + \csc^2(x+y)$$

$$\frac{dy}{dx} = \frac{\csc^2 x + \csc^2(x+y)}{-(\csc^2 y + \csc^2(x+y))}$$

$$\boxed{\frac{dy}{dx} = -\frac{\csc^2 x + \csc^2(x+y)}{\csc^2 y + \csc^2(x+y)}}$$

d) $y^2 - \csc(xy) = 0$

$$2y\frac{dy}{dx} - (-\csc xy \cot xy)\cdot\left(\frac{x\,dy}{dx} + y\right) = 0$$

$$2y\frac{dy}{dx} + \csc xy \cot xy\left(x\frac{dy}{dx} + y\right) = 0$$

$$2y\frac{dy}{dx} + x\frac{dy}{dx}\csc xy \cot xy + y\csc xy\cot xy = 0$$

$$\frac{dy}{dx}(2y + x\csc xy\cot xy) = y\csc xy\cot xy$$

$$\boxed{\frac{dy}{dx} = -\frac{y\csc xy\cot xy}{2y + x\csc xy\cot xy}}$$

e) $x^2 + \sec\left(\frac{x}{y}\right) = 0$

$$2x + \sec\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right)\left(\frac{y - \frac{x\,dy}{dx}}{y^2}\right) = 0$$

$$\frac{2x + y\sec\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right) - x\frac{dy}{dx}\sec\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right)}{y^2} = 0$$

$$2xy^2 + y\sec\left(\frac{x}{x}\right)\tan\left(\frac{x}{x}\right) - \frac{x\,dy}{dx}\sec\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right) = 0$$

$$\frac{dy}{dx} = \frac{-\left(2xy^2 + y\sec\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right)\right)}{-x\sec\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right)}$$

$$\boxed{\frac{dy}{dx} = \frac{2xy^2 + y\sec\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right)}{x\sec\left(\frac{x}{y}\right)\tan\left(\frac{x}{y}\right)}}$$

f) $y^2 = \sin(\tan y) + x^2$

$2y\frac{dy}{dx} = \cos(\tan y)\cdot \sec^2 y\,\frac{dy}{dx} + 2x$

$2y\frac{dy}{dx} - \sec^2 y\cos(\tan y)\frac{dy}{dx} = 2x$

$\frac{dy}{dx}\left(2y - \sec^2 y\cos(\tan y)\right) = 2x$

$\boxed{\frac{dy}{dx} = \frac{2x}{2y - \sec^2 y\cos(\tan y)}}$

3. Find the equations of the tangent lines.

a) $y = \cot^2 x$ when $x = \frac{\pi}{4}$

$y = (\cot x)^2$

$y' = 2\cot x \cdot -\csc^2 x$

$= -2\csc^2 x\cot x$

$= -2\frac{1}{\sin^2 x}\cdot\frac{\cos x}{\sin x}$

$y' = \frac{-2\cos x}{\sin^3 x}$ when $x = \frac{\pi}{4}$

$\frac{-2\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{1}{\sqrt{2}}\right)^3} = \frac{-\sqrt{2}}{\frac{1}{2\sqrt{2}}} \rightarrow -4$

when $x = \pi/4$ $y = \left(\frac{\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}\right)^2 = 1$

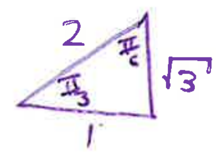
$y - 1 = -4\left(x - \frac{\pi}{4}\right)$

$y - 1 = -4x + \pi$

$\boxed{y = -4x + \pi + 1}$

b) $y = \sin x\tan\frac{x}{2}$ when $x = \frac{\pi}{3}$

$y' = \sin x\sec^2\frac{x}{2}\frac{1}{2} + \cos x\tan\frac{x}{2}$



when $x = \frac{\pi}{3}$ $\frac{1}{2}\sin\frac{\pi}{3}\cdot\left(\sec\frac{\pi}{6}\right)^2 + \cos\frac{\pi}{3}\tan\frac{\pi}{6}$

$\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\left(\frac{2}{\sqrt{3}}\right)^2 + \frac{1}{2}\cdot\frac{1}{\sqrt{3}} \rightarrow \frac{\sqrt{3}}{3} + \frac{1}{2\sqrt{3}} = \frac{2\sqrt{3}}{6} + \frac{\sqrt{3}}{6} = \frac{3\sqrt{3}}{6} \rightarrow \frac{\sqrt{3}}{2}$

at $x = \frac{\pi}{3}$

$y = \sin\frac{\pi}{3}\tan\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{3}} = \frac{1}{2}$

$\boxed{y - \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)}$

c) $y = \csc 2x$ when $x = -\frac{\pi}{8}$

$y = \csc 2(-\frac{\pi}{8}) \rightarrow \csc -\frac{\pi}{4} = \csc \frac{7\pi}{4} = -\sqrt{2}$

$y' = -\csc 2x \cot 2x \cdot 2$

$= -2\csc 2(-\frac{\pi}{8}) \cot 2(-\frac{\pi}{8})$

$= -2\csc -\frac{\pi}{4} \cot(-\frac{\pi}{4})$

$= -2\csc \frac{7\pi}{4} \cot \frac{7\pi}{4} \rightarrow -2(-\sqrt{2}) \cdot -1$

$\frac{dy}{dx} = -2\sqrt{2}$

$\boxed{y + \sqrt{2} = -2\sqrt{2}(x + \frac{\pi}{8})}$

ref angle $\frac{\pi}{4}$ Q4

d) $y = \sec x + \csc x$ when $x = \frac{3\pi}{4}$

$y = \sec \frac{3\pi}{4} + \csc \frac{3\pi}{4}$ Ref angle $\frac{\pi}{4}$ in Q2

$-\sqrt{2} + \sqrt{2} = 0$

$y = 0$

$\boxed{y = 2\sqrt{2}(x - \frac{3\pi}{4})}$

$y' = \sec x \tan x + -\csc x \cot x$

$= \sec x \tan x - \csc x \cot x$

$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} - \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$

$= \frac{\sin x}{\cos^2 x} - \frac{\cos x}{\sin^2 x} \rightarrow \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x}$ at $x = \frac{3\pi}{4}$

$\frac{\left(\frac{1}{\sqrt{2}}\right)^3 - \left(-\frac{1}{\sqrt{2}}\right)^3}{\left(\frac{1}{\sqrt{2}}\right)^2\left(-\frac{1}{\sqrt{2}}\right)^2} = \frac{\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}}{\frac{1}{4}} = 4 \cdot \frac{1}{\sqrt{2}}$

$= \frac{2 \cdot 2}{\sqrt{2}} = \boxed{2\sqrt{2}}$

4. Prove that $y = \sec x + \tan x$ is always increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y' = \sec x \tan x + \sec^2 x$

at $x = -\frac{\pi}{2}, \frac{\pi}{2}$ ← both outside interval so test any point inbetween

$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} + \frac{1}{\cos^2 x}$

y' at $x = \frac{\pi}{4}$

$= \frac{\sin x + 1}{\cos^2 x}$

$\sqrt{2} + 2 > 0$

$y' = 0$ when

$\sin x + 1 = 0$

$\sin x = -1$

$x = \frac{3\pi}{2} = -\frac{\pi}{2}$

$\boxed{\text{always increasing}}$

5. Find the vertical asymptotes.

a) $y = \csc x - \cot x,\ 0 < x < \pi$

$y = \frac{1}{\sin x} - \frac{\cos x}{\sin x}$

VA's when $\sin x = 0$

this occurs at $x = 0$, $x = \pi$ } both outside of the Domain

but $\lim_{x \to 0^+} y = 0$

$\lim_{x \to \pi^-} y = \infty$ ← this would be a VA if in the Domain

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b) $y = \sin x - \tan x,\ -\frac{\pi}{2} < x < \frac{3\pi}{2}$

$y = \sin x - \frac{\sin x}{\cos x}$

$= \frac{\sin x \cos x - \sin x}{\cos x}$

VA when $\cos x = 0$

$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ or $-\frac{\pi}{2}$

$\lim_{x \to \frac{\pi}{2}^-} y = -\infty$

$\lim_{x \to \frac{\pi}{2}^+} = \infty$

$\lim_{x \to -\frac{\pi}{2}^+} y = \infty$

$\lim_{x \to \frac{3\pi}{2}^-} y = -\infty$

6. Find the critical numbers, intervals of increase and decrease, and maximum and minimum values of $y = \csc x - \cot x$ on $(0, \pi)$

$y' = -\csc x \cot x - (-\csc^2 x)$

$= -\csc x \cot x + \csc^2 x$

$= \csc x(\csc x - \cot x)$

$y'\left(\frac{\pi}{4}\right) = \sqrt{2}\left(\sqrt{2} - 1\right)$

positive

$\boxed{\text{increasing } (0, \pi)}$

$0 = \csc x(\csc x - \cot x)$

↑ never equals zero

↓ $\frac{1}{\sin x} - \frac{\cos x}{\sin x}$

$\frac{1 - \cos x}{\sin x}$

crit pts: $x = 0$, $x = \pi$ ppc $x = 0$ } all outside domain so no max/min