## 7.3 Derivatives of Other Trigonometric Functions

Other than Sine and Cosine, we have other trigonometric functions. Derivatives of these functions can be discerned by using the Sine and Cosine identities for the other trigonometric functions.

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

### Proof

A basic identity transforms

$$y = \tan x$$

into

$$y = \frac{\sin x}{\cos x}$$

See PC 12 Trig Identities Section

And from there we can take the derivative

$$\frac{dy}{dx} = \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

### Proof

A basic identity transforms

$$y = \csc x$$

into

$$y = \frac{1}{\sin x} = (\sin x)^{-1}$$

And from there we can take the derivative

$$\frac{dy}{dx} = -(\sin x)^{-2}\frac{d}{dx}\sin x$$
$$= \frac{-1}{\sin^2 x}\cdot\cos x$$
$$= \frac{-1}{\sin x}\cdot\frac{\cos x}{\sin x}$$
$$= -\csc x\cot x$$

These proofs continue and allow us to develop a comprehensive list of derivatives for secant and cotangent functions as well. The table below summarizes these derivatives.

**Derivatives of the Trigonometric Functions**

| | |
|---|---|
| $\frac{d}{dx}\sin x = \cos x$ | $\frac{d}{dx}\csc x = -\csc x\cot x$ |
| $\frac{d}{dx}\cos x = -\sin x$ | $\frac{d}{dx}\sec x = \sec x\tan x$ |
| $\frac{d}{dx}\tan x = \sec^2 x$ | $\frac{d}{dx}\cot x = -\csc^2 x$ |

**Ex. 1** Differentiate the following

$$f(x) = \frac{1}{1+\tan x} = (1+\tan x)^{-1}$$

$$f'(x) = -1(1+\tan x)^{-2}\cdot\sec^2 x$$

$$= \boxed{\frac{-\sec^2 x}{(1+\tan x)^2}}$$

there are more trig identities we could use to acquire a simplified form

$$y' = \frac{-1}{1+\sin 2x}$$

$$\frac{-\sec^2 x}{1+2\tan x+\tan^2 x}$$

$$\frac{-\sec^2 x}{2\tan x+1+\tan^2 x}$$

$$\frac{-\sec^2 x}{2\tan x+\sec^2 x}$$

$$\frac{-\sec^2 x}{\frac{2\sin x}{\cos x}+\frac{1}{\cos^2 x}}$$

$$\frac{-\sec^2 x}{\frac{2\sin x\cos x+1}{\cos^2 x}} \rightarrow \frac{-\sec^2 x\cdot\cos^2 x}{2\sin x\cos x+1}$$

$$= \frac{-\frac{1}{\cos^2 x}\cdot\cos^2 x}{1+\sin 2x} =$$

**Ex. 2** Differentiate the following

$$y = 2\csc^3(3x^2) \rightarrow y = 2(\csc(3x^2))^3$$

$$y' = 6(\csc(3x^2))^2 \cdot -\csc(3x^2)\cot(3x^2) \cdot 6x$$

$$= -6 \cdot 6x \cdot \csc(3x^2)\cot(3x^2)\csc^2(3x^2)$$

$$= \boxed{-36x\csc^3(3x^2)\cot(3x^2)}$$

**Ex. 3** If $\tan y = x^2$ find the derivative of $y$ with respect to $x$

Implicit Differentiation

$$\frac{d}{dx}(\tan y = x^2)$$

$$\sec^2 y \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = 2x \cdot \frac{1}{\sec^2 y}$$

$$= 2x \cdot \cos^2 y \rightarrow \boxed{2x\cos^2 y}$$

Q1

**Ex. 4** Find the slope of the tangent line to $y = \tan(\csc x)$ when $\sin x = 1/\pi$ and $x$ is in the interval $\left(0, \frac{\pi}{2}\right)$

$$y = \tan(\csc x)$$

$$y' = \sec^2(\csc x) \cdot -\csc x \cot x$$

$$y' = \sec^2(\pi)\left[-\left(\pi \cdot \sqrt{\pi^2 - 1}\right)\right]$$

$$= (\sec\pi)^2\left[-\pi\sqrt{\pi^2-1}\right]$$

$$= \left(\frac{1}{\cos\pi}\right)^2\left[-\pi\sqrt{\pi^2-1}\right]$$

$$(-1)^2\left[-\pi\sqrt{\pi^2-1}\right]$$

$$\boxed{-\pi\sqrt{\pi^2-1}}$$

if $\sin x = \frac{1}{\pi}$

$\csc x = \pi$

if $\csc x = \pi$

$x^2 = r^2 - y^2 \rightarrow x^2 = \pi^2 - 1^2$

$x = \sqrt{\pi^2 - 1}$

$y = 1$

$r = \pi$

$\cot = \frac{x}{y} = \sqrt{\pi^2 - 1}$

$\csc x = \frac{r}{y}$

**Ex. 5** Prove that $y = \sec x + \tan x$ is concave up on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y' = \sec x \tan x + \sec^2 x$

$= \sec x(\tan x + \sec x)$

$y'' = \sec x(\sec^2 x + \sec x \tan x) + (\tan x + \sec x)(\sec x \tan x)$

$= \sec^2 x(\sec x + \tan x) + \tan^2 x \sec x + \sec^2 x \tan x$

$= \sec^3 x + \sec^2 x \tan x + \sec^2 x \tan x + \tan^2 x \sec x$

$= \sec^3 x + 2\sec^2 x \tan x + \tan^2 x \sec x$

$= \sec x(\sec^2 x + 2\sec x \tan x + \tan^2 x)$

$= \sec x(\sec x + \tan x)^2$

if interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

pick $x = 0$

$\sec 0(\sec 0 + \tan 0)^2$

$1(1+0) = 1$ (pos)

cu

**Ex. 6** Find the vertical asymptotes of $y = \sec x + \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and provide a rough sketch of the function.

$y = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1+\sin x}{\cos x}$

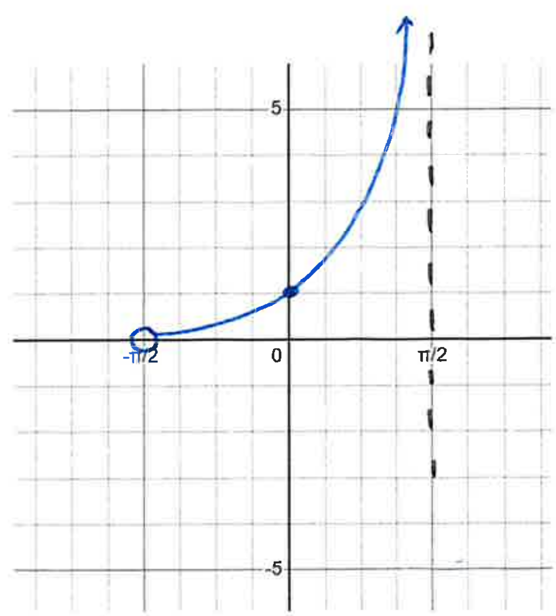
on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\cos x = 0$ when $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$ (these are possibilities)

$\lim_{x \to -\frac{\pi}{2}^+} \frac{1+\sin x}{\cos x}$ ← radicalize. $\frac{1-\sin^2 x}{(1-\sin x)\cos x} = \frac{\cos^2 x}{\cos x(1-\sin x)} = \frac{\cos x}{1-\sin x}$

$\lim_{x \to -\frac{\pi}{2}^+} \frac{0}{1-(-1)} = 0$

$\lim_{x \to \frac{\pi}{2}^-} \frac{1+\sin x}{\cos x} \to \frac{\text{very large}}{\text{very small}} \to \infty$




**Homework Assignment**

- Practice Problems: 1acehjlo, 2adef, 3ac,4-6