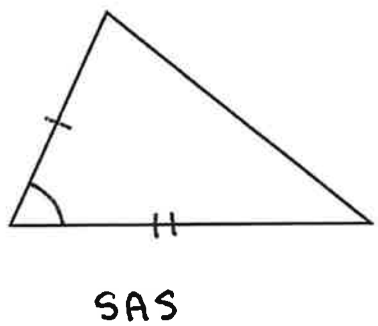
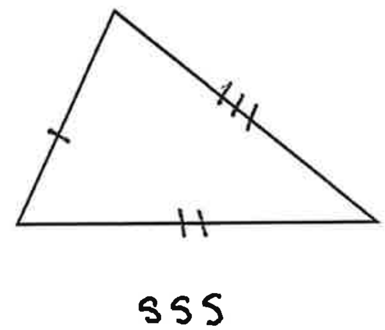
Section 7.3 – The Law of Cosines

- Use the Law of Cosines when you know two sides and a contained angle (SAS) or when you know all three sides (SSS):





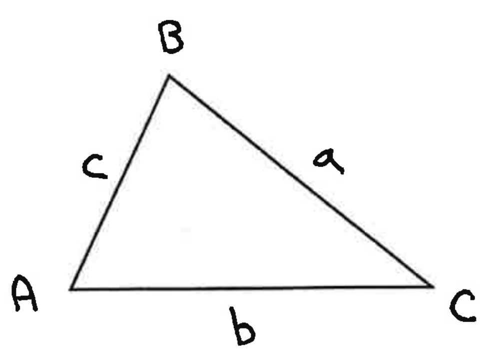
The Law of Cosines

For any triangle ABC with correspomding sides $a, b, and\ c$:

$$a^2 = b^2 + c^2 - 2bcCosA \qquad Cos\ A = \frac{b^2 + c^2 - a^2}{2bc}$$

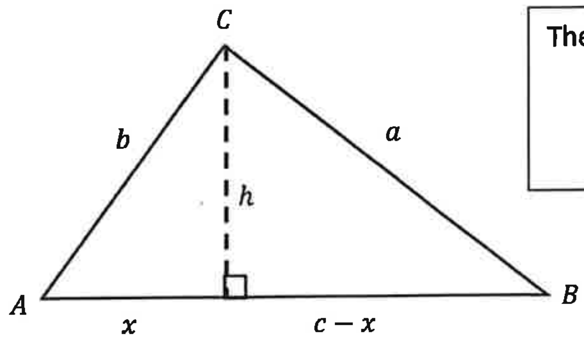
$$b^2 = a^2 + c^2 - 2acCosB \qquad Cos\ B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2abCosC \qquad Cos\ C = \frac{a^2 + b^2 - c^2}{2ab}$$



Derivation

- **Consider the Oblique triangle ABC**



The length c, is divided into two parts: x and $c - x$

$$Cos\,A = \frac{x}{b} \quad \rightarrow \quad x = bCosA$$

By Pythagorean Theorem:

$$b^2 = h^2 + x^2 \quad \rightarrow \quad h^2 = b^2 - x^2$$

$$a^2 = h^2 + (c-x)^2 \quad \rightarrow \quad h^2 = a^2 - (c-x)^2$$

By setting h^2 *equal to one another*:

$$a^2 - (c-x)^2 = b^2 - x^2$$

$$\rightarrow \quad a^2 = b^2 - x^2 + (c-x)^2$$

$$\rightarrow \quad a^2 = b^2 - x^2 + (c-x)(c-x)$$

$$\rightarrow \quad a^2 = b^2 - x^2 + c^2 - 2cx + x^2$$

$$\rightarrow \quad a^2 = b^2 + c^2 - 2cx$$

$$\rightarrow \quad a^2 = b^2 + c^2 - 2bcCosA$$

Using the Law of Cosines for SSS

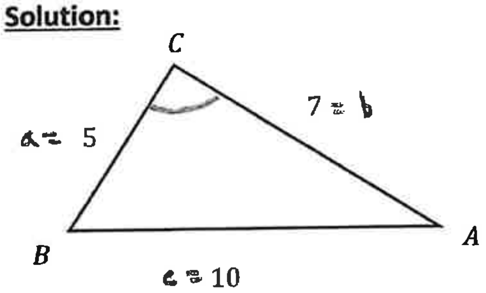
- When you have a SSS triangle **ALWAYS** find the **largest angle first**. This will **guarantee** that the other two angles are acute and you can use the easier Law of Sines to solve.
- When you have a SAS triangle, after you solve for length of side, use the Law of Sines **ALWAYS** on the **Smallest Angle** next. The Law of Sines will not pick up on an obtuse angle.

Using the Law of Cosines for SSS

- When you have a SSS triangle **ALWAYS** find the **largest angle first**. (opposite largest side)
- This will **guarantee** that the other two angles are **ACUTE**
- There is **NO AMBIGUOUS CASE** for the LAW of Cosines (THANK YOU!!)

Example: Solve $\Delta ABC, given\ a = 5, b = 7, and\ c = 10$

Solution:



$$c^2 = a^2 + b^2 - 2abCosC$$

$$10^2 = 5^2 + 7^2 - 2(5)(7)\cos C$$

$$100 = 25 + 49 - 70\cos C$$

$$100 = 74 - 70\cos C$$

$$26 = -70\cos C$$

$$\frac{26}{-70} = \cos C$$

$$\angle C = \cos^{-1}\left(-\frac{26}{70}\right)$$

$$= 111.8^\circ$$

- Now we can use the LAW of SINES to find one of the other two angles.

$$\frac{Sin\ A}{a} = \frac{Sin\ C}{c}$$

$$\frac{\sin A}{5} = \frac{\sin 111.8^\circ}{10}$$

$$\sin A = \frac{5\sin 111.8^\circ}{10}$$

$$\angle A = \sin^{-1}\left(\frac{5\sin 111.8^\circ}{10}\right)$$

$$\angle A = 27.7^\circ$$

So, $\angle B = 180° - 27.7° - 111.8°$

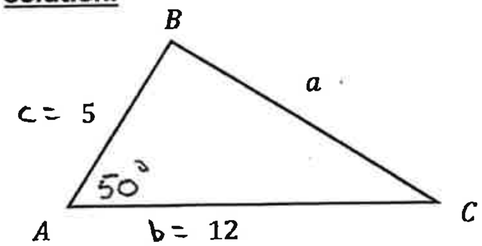
$$\angle B = 40.5^\circ$$

Note: If we had solved for another angle first we would have gotten the **WRONG** solution. **ALWAYS** solve the **LARGEST ANGLE FIRST** in a **SSS** problem!

Using the Law of Cosines for SAS

Example: Solve $\Delta ABC, given\ \angle A = 50°, b = 12, and\ c = 5$

Solution:



$$a^2 = b^2 + c^2 - 2bcCosA$$

$$a^2 = 12^2 + 5^2 - 2(12)(5)\cos 50°$$

$$a^2 = 144 + 25 - 120\cos 50°$$

$$a^2 = 169 - 77.135$$

$$a^2 = 91.865$$

$$a = \sqrt{91.865}$$

$$a = 9.58$$

- Now we can use the LAW of SINES to find one of the other two angles.

$$\frac{Sin\ A}{a} = \frac{Sin\ C}{c}$$

$$\frac{\sin 50°}{9.58} = \frac{\sin C}{5}$$

$$\sin C = \frac{5\sin 50°}{9.58}$$

$$\angle C = \sin^{-1}\left(\frac{5\sin 50°}{9.58}\right)$$

$$= 23.6°$$

So, $\angle B = 180° - 23.6° - 50°$

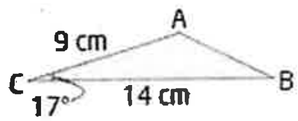
$$= 106.4°$$

Note: If we had solved for another angle first we would have gotten the **WRONG** solution. <u>**ALWAYS**</u> solve the **SMALLEST ANGLE FIRST** in a **SAS** problem!

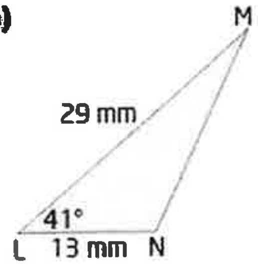
Section 7.3 – Practice Questions

1. Determine the length of the third side of each triangle

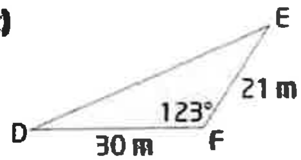
a)



b)

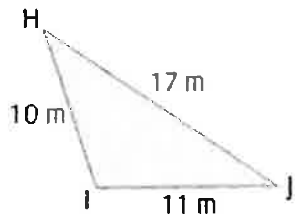


c)

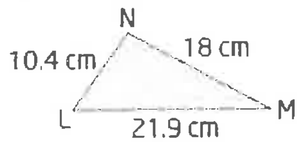


2. Determine the measure of the indicated angle

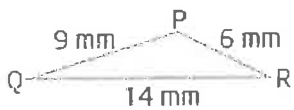
a) $\angle J$



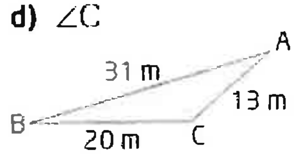
b) $\angle L$



c) $\angle P$

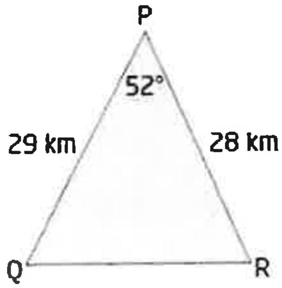


d) $\angle C$

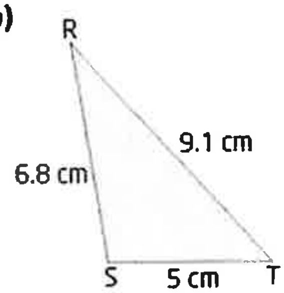


3. Determine the lengths of the unknown sides and the measures of the unknown angles.

a)



b)



4. Solve triangle ABC. Round answers to one decimal place.

i) $\angle A = 50°, b = 10, c = 15$

ii) $\angle B = 36°, a = 4, c = 10$

iii) $\angle C = 60°, b = 4, a = 8$

iv) $a = 7, b = 24, c = 25$

v) $a = 6, b = 7, c = 8$

vi) $\angle A = 120°, b = 4, c = 1$

5. An aircraft tracking station determines the distances from a helicopter to two aircraft as 50 km and 72 km. The angle between these two distances is 49°. Determine the distance between the two aircraft.