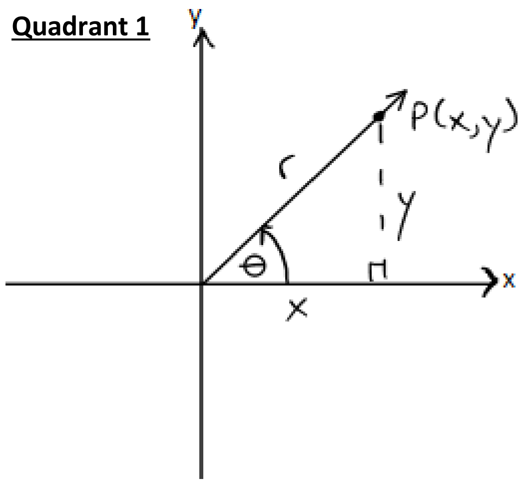
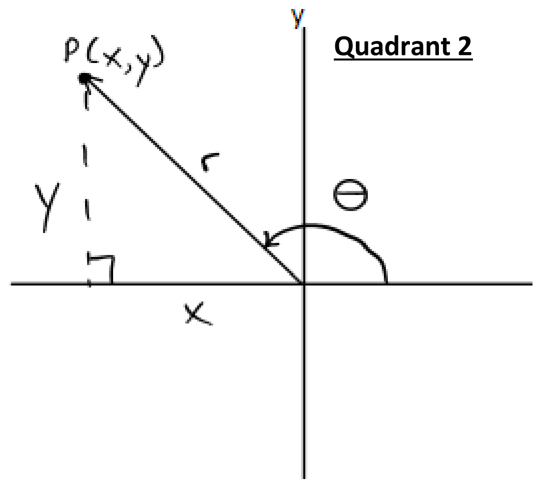
Section 7.2 – The Three Trigonometric Ratios

- This is where we see now trigonometric functions tie into the SOH CAH TOA we learned in grade 10. We need to adjust how we look at triangles created within the coordinate plane
- Consider an **angle created by the terminal arm**, with **radius r**. Let **θ be the angle** in **Standard Position**, and the **point P with coordinates** (x, y).
- We have a scenario in each Quadrant

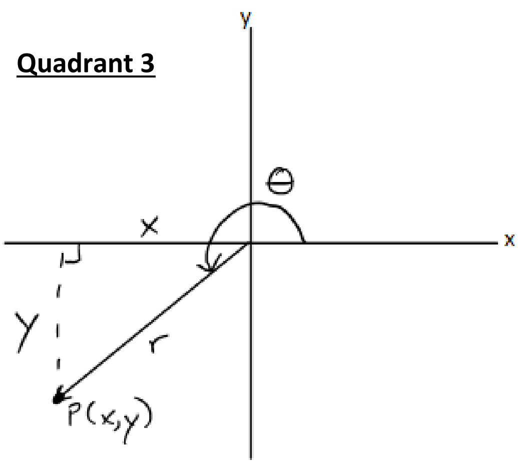
Quadrant 1



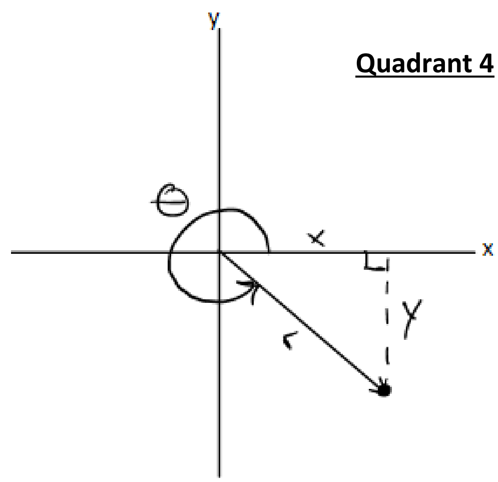
Quadrant 2



Quadrant 3



Quadrant 4



- By the Pythagorean Theorem, $\boldsymbol{r^2 = x^2 + y^2}$. The distance from the point $(0, 0)$ to (x, y) is $\boldsymbol{r}$ and is therefore **always positive!** $\boldsymbol{r = \sqrt{x^2 + y^2}}$
- The values of x, y and r determine the three trigonometric ratios for angle θ. And this is where we see: $\sin\theta, \cos\theta, and \tan\theta$. Sine, Cosine, and Tangent.

Trigonometric Ratios

- As you can see from the image above, **if $\boldsymbol{\theta}$ is in Standard Position** with point $P(x, y)$ on the terminal arm, then the trigonometric ratios SOH CAH TOA are now defined as:

$$\boldsymbol{\sin\theta = \frac{y}{r} \quad \left(\frac{opposite}{hypotenuse}\right) \qquad \cos\theta = \frac{x}{r} \quad \left(\frac{adjacent}{hypotenuse}\right) \qquad \tan\theta = \frac{y}{x},\ \ x \neq 0 \quad \left(\frac{opposite}{adjacent}\right)}$$

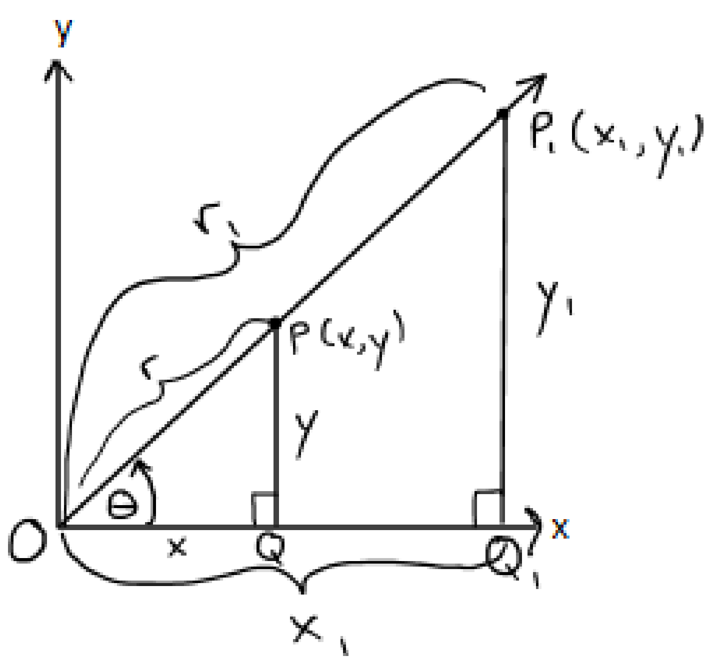
$$y = the\ side\ opposite\ \theta$$

$$x = the\ side\ adjacent\ \theta$$

$$r = the\ length\ of\ the\ radius\ (hypotenuse)\ with\ r = \sqrt{x^2 + y^2}$$

- The value of the trigonometric ratio is determined by the angle θ, not the lengths of x, y
- Think proportional triangles

Example:



Since ΔOPQ is similar to ΔOP_1Q_1, the corresponding sides are proportionate.

$$\sin\theta = \frac{y}{r}\ and\ \sin\theta = \frac{y_1}{r_1}, so\ \frac{y}{r} = \frac{y_1}{r_1}$$

$$\cos\theta = \frac{x}{r}\ and\ \cos\theta = \frac{x_1}{r_1}, so\ \frac{x}{r} = \frac{x_1}{r_1}$$

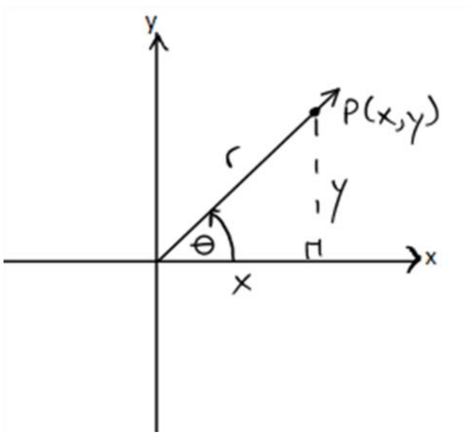
$$\tan\theta = \frac{y}{x}\ and\ \sin\theta = \frac{y_1}{x_1}, so\ \frac{y}{x} = \frac{y_1}{x_1}$$

The **value of each ratio depends on the angle $\boldsymbol{\theta}$**, not the lengths of $x, y, and\ r$, which change depending on which point on the terminal arm is used. By the sides change proportionally.

Sign Properties of the Trigonometric Functions

- Depending on the trig function you are solving for, there are specific patterns as to the sign of the ratio you are solving for.
- When considering the triangles that are formed by **the reference angle**, taking into consideration the **quadrant you are in**, the sign will change depending on the ratio being used.

Quadrant 1



For θ in $Q1$: $x > 0, y > 0$

Therefore:

$$\sin\theta = \frac{y}{r} = \frac{+}{+} = +$$

$$\cos\theta = \frac{x}{r} = \frac{+}{+} = +$$

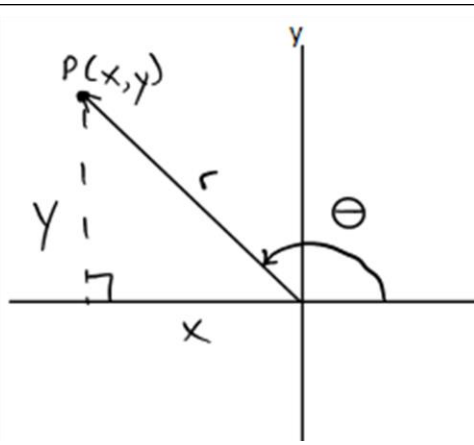
$$\tan\theta = \frac{y}{x} = \frac{+}{+} = +$$

All three trigonometric functions are positive when θ is in Q1

Example: For the point $P(3,4)$ in $Q1$, the value of $r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, therefore

$$\sin\theta = \frac{y}{r} = \frac{4}{5} \qquad \cos\theta = \frac{x}{r} = \frac{3}{5} \qquad \tan\theta = \frac{y}{x} = \frac{4}{3}$$

Quadrant 2



For θ in $Q2$: $x < 0, y > 0$

Therefore:

$$\sin\theta = \frac{y}{r} = \frac{+}{+} = +$$

$$\cos\theta = \frac{x}{r} = \frac{-}{+} = -$$

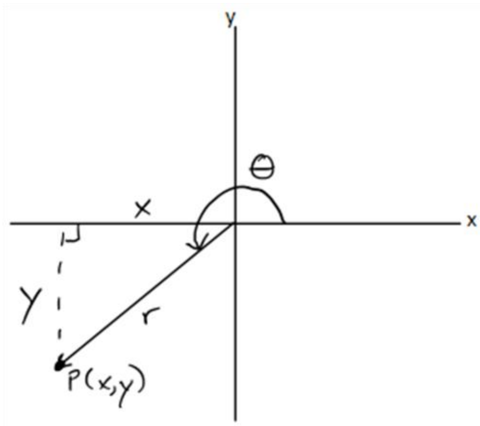
$$\tan\theta = \frac{y}{x} = \frac{+}{-} = -$$

$\sin\theta$ is positive when θ is in Q2
$\cos\theta$ *and* $\tan\theta$ are negative in Q2

Example: For the point $P(-3,4)$ in $Q2$, the value of $r = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$, therefore

$$\sin\theta = \frac{y}{r} = \frac{4}{5} \qquad \cos\theta = \frac{x}{r} = -\frac{3}{5} \qquad \tan\theta = \frac{y}{x} = -\frac{4}{3}$$

Quadrant 3



For θ in $Q3$: $x < 0, y < 0$

Therefore:

$$\sin\theta = \frac{y}{r} = \frac{-}{+} = -$$

$$\cos\theta = \frac{x}{r} = \frac{-}{+} = -$$

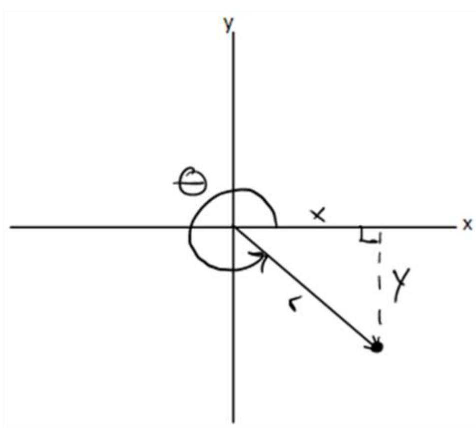
$$\tan\theta = \frac{y}{x} = \frac{-}{-} = +$$

$\tan\theta$ is positive when θ is in Q3
$\cos\theta$ *and* $\sin\theta$ are negative in Q3

Example: For the point $P(-3,-4)$ in $Q3$, the value of $r = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$, therefore

$$\sin\theta = \frac{y}{r} = -\frac{4}{5} \qquad \cos\theta = \frac{x}{r} = -\frac{3}{5} \qquad \tan\theta = \frac{y}{x} = \frac{-4}{-3} = \frac{4}{3}$$

Quadrant 4



For θ in $Q4$: $x > 0, y < 0$

Therefore:

$$\sin\theta = \frac{y}{r} = \frac{-}{+} = -$$

$$\cos\theta = \frac{x}{r} = \frac{+}{+} = +$$

$$\tan\theta = \frac{y}{x} = \frac{-}{+} = -$$

$\cos\theta$ is positive when θ is in Q4
$\sin\theta$ *and* $\tan\theta$ are negative in Q4

Example: For the point $P(3,-4)$ in $Q4$, the value of $r = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$, therefore

$$\sin\theta = \frac{y}{r} = -\frac{4}{5} \qquad \cos\theta = \frac{x}{r} = \frac{3}{5} \qquad \tan\theta = \frac{y}{x} = -\frac{4}{3}$$

Summary

Quadrant (θ)	Positive Function	Negative Function
1	*All*	*None*
2	$\sin\theta$	$\cos\theta, \tan\theta$
3	$\tan\theta$	$\sin\theta, \cos\theta$
4	$\cos\theta$	$\sin\theta, \tan\theta$

Positive Where?: **All Students Take Calculus**

2 $\sin\theta$	1 *All*
3 $\tan\theta$	4 $\cos\theta$

Example 1: Identify the quadrant(s) for the angles satisfying the following conditions

a) $\sin\theta < 0, \cos\theta > 0$ b) $\tan\theta < 0, \cos\theta < 0$

Solution 1:

a) $\sin\theta$ is negative in $Q3$ *and* $\boldsymbol{Q4}$; $\cos\theta$ is positive in $Q1$ *and* $\boldsymbol{Q4}$ so.. **$\boldsymbol{Q4}$ satisfies both**

b) $\tan\theta$ is negative in $\boldsymbol{Q2}$ *and* $Q4$; $\cos\theta$ is negative in $\boldsymbol{Q2}$ *and* $Q3$ so.. **$\boldsymbol{Q2}$ satisfies both**

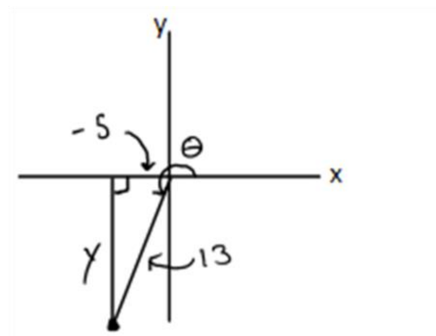
Example 2: If $\cos\theta = -\frac{5}{13}$ with θ in $Q3$, find $\sin\theta$ *and* $\tan\theta$.

Solution 2: $\cos\theta = -\frac{5}{13}$, so the $x - value\ is -5, and\ the\ radius\ is\ 13\ (can'tbe\ negative)$

$$x^2 + y^2 = r^2$$
$$(-5)^2 + y^2 = 13^2$$
$$25 + y^2 = 169$$
$$y^2 = 144$$
$$y = \pm\sqrt{144} = \pm 12$$

In $\boldsymbol{Q3}$ so $\boldsymbol{y < 0}$ therefore $\boldsymbol{y = -12}$

$$\boldsymbol{\sin\theta = \frac{y}{r} = -\frac{12}{13}};\ \boldsymbol{\tan\theta = \frac{y}{x} = \frac{-12}{-5} = \frac{12}{5}}$$



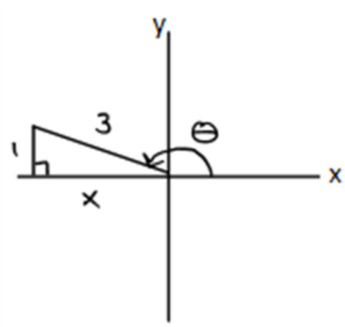
Example 3: Find $\tan\theta$ if $\sin\theta = \frac{1}{3}$ with θ in $Q2$

Solution 3: $\sin\theta = \frac{1}{3}$, so the $y - value\ is\ 1, and\ the\ radius\ is\ 3$

$$x^2 + y^2 = r^2$$
$$x^2 + 1^2 = 3^2$$
$$x^2 + 1 = 9$$
$$x^2 = 8$$
$$x = \pm\sqrt{8} = \pm 2\sqrt{2}$$

In $\boldsymbol{Q2}$ so $\boldsymbol{x < 0}$ therefore $\boldsymbol{x = -2\sqrt{2}}$

$$\boldsymbol{\tan\theta = \frac{y}{x} = \frac{1}{-2\sqrt{2}} = -\frac{\sqrt{2}}{4}}$$



Example 4: Find $\sin\theta$ if $\tan\theta = 2.135$ with θ in $Q3$

Solution 4: $\tan\theta = \frac{y}{x}$, and in $Q3$ both x and y are $negative$ so $\tan\theta = \frac{-2.135}{-1}$

$$x^2 + y^2 = r^2$$

$$(-1)^2 + (-2.135)^2 = r^2$$

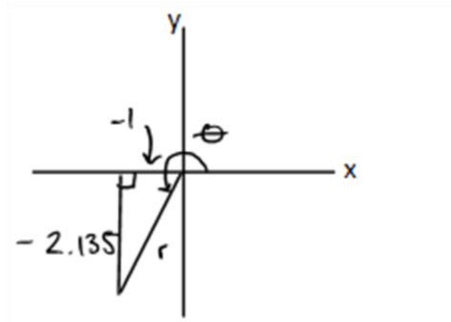
$$r^2 = 1 + 4.558$$

$$r^2 = 5.558$$

$$r = \sqrt{5.558} = 2.358$$

***r* can only be positive**

$$\boldsymbol{\sin\theta = \frac{y}{r} = \frac{-2.135}{2.358} = -0.905}$$



Example 5: $y = -2x, x \le 0$ is the equation of the terminal side of an angle θ in Standard Position. Sketch the smallest positive angle θ, and determine $\sin\theta, \cos\theta, and \tan\theta$.

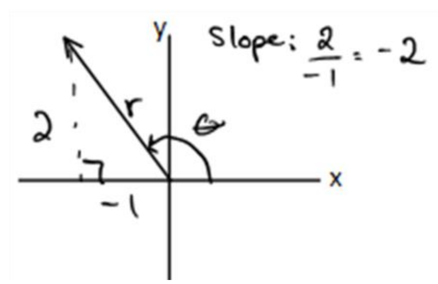
Solution 5: With $x \le 0$ the Slope of $y = -2x$ is: $\frac{Rise}{Run} = \frac{y}{x} = \frac{2}{-1}$. $(-1, 2)$ is on the terminal arm

$$x^2 + y^2 = r^2$$

$$(-1)^2 + 2^2 = r^2$$

$$r = \sqrt{5}$$

$$\boldsymbol{\sin\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5};\ \cos\theta = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5};\ \tan\theta = \frac{2}{-1} = -2}$$



Example 6: $y = 3x, x \le 0$ is the equation of the terminal side of an angle θ in Standard Position. Sketch the smallest positive angle θ, and determine $\sin\theta, \cos\theta, and \tan\theta$.

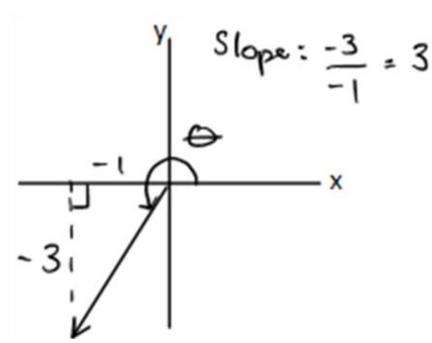
Solution 6: With $x \le 0$ the Slope of $y = 3x$ is: $\frac{Rise}{Run} = \frac{y}{x} = \frac{-3}{-1}$. $(-1, -3)$ is on the terminal arm

$$x^2 + y^2 = r^2$$

$$(-1)^2 + (-3)^2 = r^2$$

$$r = \sqrt{10}$$

$$\boldsymbol{\sin\theta = -\frac{3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10};\ \cos\theta = -\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{10};\ \tan\theta = \frac{-3}{-1} = 3}$$



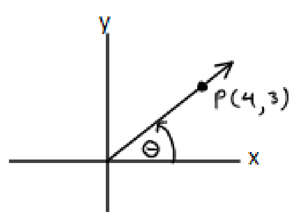
Section 7.2 – Practice Problems

If the following angles satisfy the given conditions, what quadrant are they found in?

1. $\cos\theta < 0$	2. $\tan\theta > 0$
3. $\sin\theta < 0$	4. $\sin\theta > 0$ and $\tan\theta < 0$
5. $\cos\theta < 0$ *and* $\tan\theta > 0$	6. $\sin\theta <$ and $\cos\theta > 0$
7. $\sin\theta > 0$ *and* $\tan\theta > 0$	8. $\cos\theta < 0$ *and* $\tan\theta < 0$
9. $\sin\theta < 0$ and $\tan\theta < 0$	10. $\sin < 0$ and $\cos\theta < 0$

Given is a point on the terminal side of θ is shown. Evaluate the three trigonometric functions of θ

11.



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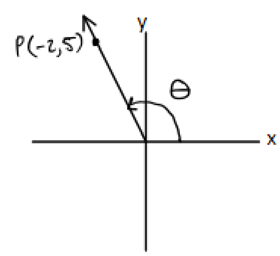

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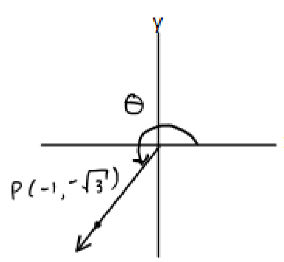

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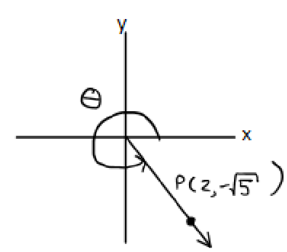

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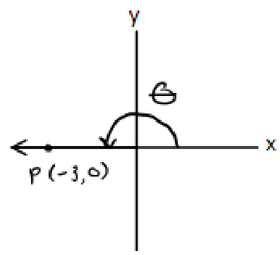

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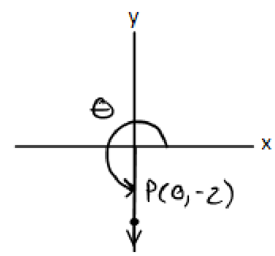

If θ is in Standard Position and the given point is on the terminal side of θ, find the values of the three trigonometric functions of θ

17. $(3, -4)$

18. $(-12, 5)$

19. $(-7, -24)$

20. $(8, 15)$

21. $(-2\sqrt{3}, 2)$

22. $(\sqrt{2}, \sqrt{7})$

23. $(-3, -3)$

24. $(0, 4)$

25. $(-2, 0)$

26. $(-\sqrt{5}, 2)$

27. $(-5, 5)$

28. $(-2\sqrt{2}, 1)$

29. $(-40, -9)$

30. $(9, -40)$

31. $(2\sqrt{2}, 8)$

32. $(-3, \sqrt{3})$

Given is one of the trigonometric functions and some extra information. Use the information to find the other two trigonometric functions of the angle.

33. $\sin\theta = \frac{4}{5}$; $\theta\ is\ in\ Q1$

34. $\cos\theta = -\frac{12}{13}$; $\theta\ is\ in\ Q2$

35. $\tan\theta = \frac{7}{24}$; $\theta\ is\ in\ Q3$

36. $\sin\theta = -\frac{\sqrt{3}}{2}$; $\theta\ is\ in\ Q4$

37. $\cos\theta = \frac{1}{2}$; $\tan\theta > 0$

38. $\tan\theta = \frac{2}{\sqrt{5}}$; $\sin\theta > 0$

39. $\sin\theta = -\frac{3}{\sqrt{10}}$; $\cos\theta < 0$

40. $cos\,\frac{3}{\sqrt{13}}$; $\tan\theta < 0$

41. $\tan\theta = -1$; $\sin\theta < 0$

42. $\sin\theta = \frac{\sqrt{15}}{4}$

Given is the value of one of the trigonometric functions and some extra information. Use the information to find the other two trigonometric functions of the angle. Round answers to three decimal places

43. $\sin\theta = 0.642$; $\theta\ is\ in\ Q1$

44. $\cos\theta = 0.537$; $\theta\ is\ in\ Q4$

45. $\tan\theta = 2\,;\ \theta\ is\ in\ Q3$

46. $\sin\theta = 0.237\,;\ \theta\ is\ in\ Q2$

47. $\cos\theta = -0.378\,; \sin\theta > 0$

48. $\tan\theta = -1.413\,; \cos\theta > 0$

49. $\sin\theta = -0.753\,; \tan\theta > 0$

50. $\cos\theta = -0.492\,; \sin\theta > 0$

Answer Key – Section 7.2

1. $Q2\ and\ Q3$
2. $Q1\ and\ Q3$
3. $Q3\ and\ Q4$
4. $Q2$
5. $Q3$
6. $Q4$
7. $Q1$
8. $Q2$
9. $Q4$
10. $Q3$
11. $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$
12. $\sin\theta = \frac{5}{\sqrt{29}}$, $\cos\theta = -\frac{2}{\sqrt{29}}$, $\tan\theta = -\frac{5}{2}$
13. $\sin\theta = -\frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$, $\tan\theta = \sqrt{3}$
14. $\sin\theta = -\frac{\sqrt{5}}{3}$, $\cos\theta = \frac{2}{3}$, $\tan\theta = -\frac{\sqrt{5}}{2}$
15. $\sin\theta = 0$, $\cos\theta = -1$, $\tan\theta = 0$
16. $\sin\theta = -1$, $\cos\theta = 0$, $\tan\theta = Undefined$
17. $\sin\theta = -\frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = -\frac{4}{3}$
18. $\sin\theta = \frac{5}{13}$, $\cos\theta = -\frac{12}{13}$, $\tan\theta = -\frac{5}{12}$
19. $\sin\theta = -\frac{24}{25}$, $\cos\theta = -\frac{7}{25}$, $\tan\theta = \frac{24}{7}$
20. $\sin\theta = \frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = \frac{15}{8}$
21. $\sin\theta = \frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$, $\tan\theta = -\frac{1}{\sqrt{3}}\ or\ -\frac{\sqrt{3}}{3}$
22. $\sin\theta = \frac{\sqrt{7}}{3}$, $\cos\theta = \frac{\sqrt{2}}{3}$, $\tan\theta = \frac{\sqrt{7}}{\sqrt{2}}\ or\ \frac{\sqrt{14}}{2}$
23. $\sin\theta = -\frac{1}{\sqrt{2}}\ or\ -\frac{\sqrt{2}}{2}$, $\cos\theta = -\frac{1}{\sqrt{2}}\ or\ -\frac{\sqrt{2}}{2}$, $\tan\theta = 1$
24. $\sin\theta = 1$, $\cos\theta = 0$, $\tan\theta = Undefined$
25. $\sin\theta = 0$, $\cos\theta = -1$, $\tan\theta = 0$
26. $\sin\theta = \frac{2}{3}$, $\cos\theta = -\frac{\sqrt{5}}{3}$, $\tan\theta = -\frac{2}{\sqrt{5}}\ or\ -\frac{2\sqrt{5}}{5}$
27. $\sin\theta = \frac{1}{\sqrt{2}}\ or\ \frac{\sqrt{2}}{2}$, $\cos\theta = -\frac{1}{\sqrt{2}}\ or\ -\frac{\sqrt{2}}{2}$, $\tan\theta = -1$
28. $\sin\theta = \frac{1}{3}$, $\cos\theta = -\frac{2\sqrt{2}}{3}$, $\tan\theta = -\frac{1}{2\sqrt{2}}\ or\ -\frac{\sqrt{2}}{4}$
29. $\sin\theta = -\frac{9}{41}$, $\cos\theta = -\frac{40}{41}$, $\tan\theta = \frac{9}{40}$
30. $\sin\theta = -\frac{40}{41}$, $\cos\theta = \frac{9}{41}$, $\tan\theta = -\frac{40}{9}$
31. $\sin\theta = \frac{2\sqrt{2}}{3}$, $\cos\theta = \frac{1}{3}$, $\tan\theta = 2\sqrt{2}$

32. $\sin\theta = \frac{1}{2}$
$\cos\theta = -\frac{\sqrt{3}}{2}$
$\tan\theta = -\frac{\sqrt{3}}{3}$

33. $\cos\theta = \frac{3}{5}$
$\tan\theta = \frac{4}{3}$

34. $\sin\theta = \frac{5}{13}$
$\tan\theta = -\frac{5}{12}$

35. $\sin\theta = -\frac{7}{25}$
$\cos\theta = -\frac{24}{25}$

36. $\cos\theta = \frac{1}{2}$
$\tan\theta = -\sqrt{3}$

37. $\sin\theta = \frac{\sqrt{3}}{2}$
$\tan\theta = \sqrt{3}$

38. $\sin\theta = \frac{2}{3}$
$\cos\theta = \frac{\sqrt{5}}{3}$

39. $\cos\theta = -\frac{1}{\sqrt{10}}$
$\tan\theta = 3$

40. $\sin\theta = -\frac{2}{\sqrt{13}}$
$\tan\theta = -\frac{2}{3}$

41. $\sin\theta = -\frac{1}{\sqrt{2}}$
$\cos\theta = \frac{1}{\sqrt{2}}$

42. $\cos\theta = \pm\frac{1}{4}$
$\tan\theta = \pm\sqrt{15}$

43. $\cos\theta = 0.767$
$\tan\theta = 0.837$

44. $\sin\theta = -0.844$
$\tan\theta = -1.571$

45. $\sin\theta = -0.894$
$\cos\theta = -0.447$

46. $\cos\theta = -0.972$
$\tan\theta = -0.244$

47. $\sin\theta = 0.926$
$\tan\theta = -2.450$

48. $\sin\theta = -0.816$
$\cos\theta = 0.578$

49. $\cos\theta = -0.658$
$\tan\theta = 1.144$

50. $\sin\theta = 0.871$
$\tan\theta = -1.770$

Extra Work Space