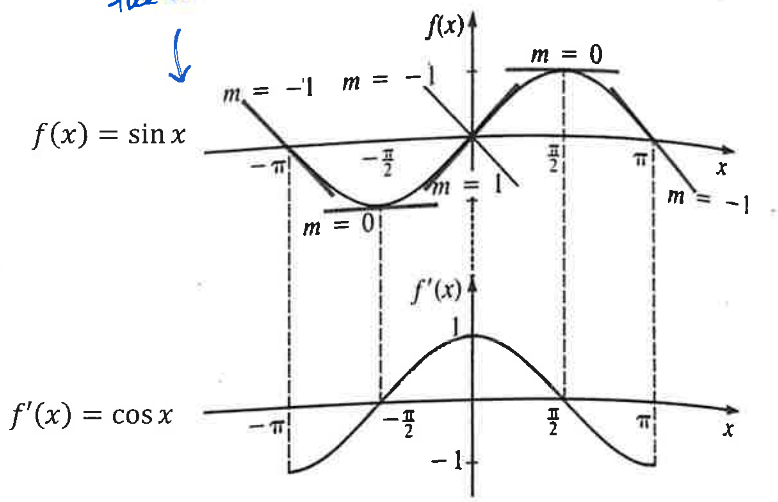
7.2 Derivatives of the Sine and Cosine Functions

If we apply the interpretation of $f'(x)$ as the slope of the tangent line at $(x, f(x))$ to the functions $f(x) = \sin x$, it appears that $f'(x) = \cos x$. That is,

$$\boxed{\frac{d}{dx}\sin x = \cos x}$$

attach to the visual



Proof

$$f(x) = \sin x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x\,(\cos h - 1) + \cos x \sin h}{h}$$

$$= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

proved in 7.1
do not stress.

$$= \sin x(0) + \cos(1)$$

$$= \cos x$$

We use some concepts previously discussed in Section 7.1 to aid with the proof above.

$$\boxed{\frac{d}{dx}\cos x = -\sin x}$$

Proof

$$f(x) = \cos x$$

$$f(x) = \sin\left(\frac{\pi}{2} - x\right)$$

$$f'(x) = \cos\left(\frac{\pi}{2} - x\right)\frac{d}{dx}\left(\frac{\pi}{2} - x\right)$$

$$= (\sin x)(-1)$$

$$= -\sin x$$

Ex. 1 Differentiate

a) $y = \sin 3x$

$$\frac{dy}{dx} = \cos 3x \frac{d}{dx}3x$$

$$= \cos 3x \cdot 3$$

$$= \boxed{3\cos 3x}$$

b) $y = \sin(x + 2)$

$$\frac{dy}{dx} = \cos(x+2)\frac{d}{dx}(x+2)$$

$$= \boxed{\cos(x+2)}$$

c) $y = \sin(kx + d)$

$$\frac{dy}{dx} = \cos(kx+d)\frac{d}{dx}(kx+d)$$

$$= \cos(kx+d)\cdot k$$

$$= \boxed{k\cos(kx+d)}$$

Ex. 2 Differentiate

a) $y = \sin(x^3)$

$$\frac{dy}{dx} = \cos(x^3) \cdot 3x^2$$

$$= \boxed{3x^2 \cos(x^3)}$$

b) $y = \sin^3 x \rightarrow (\sin x)^3$

$$\frac{dy}{dx} = 3(\sin x)^2 \cdot \cos x$$

$$= \boxed{3\sin^2 x \cos x}$$

c) $y = \sin^3(x^2 - 1) \rightarrow (\sin(x^2-1))^3$

$$\frac{dy}{dx} = 3(\sin(x^2-1))^2 \cdot \cos(x^2-1) \cdot 2x$$

$$= \boxed{6x\cos(x^2-1)\sin^2(x^2-1)}$$

Ex. 3 Differentiate $y = x^2 \cos x$ (Product Rule)

$$\frac{dy}{dx} = x^2(-\sin x) + 2x\cos x$$

$$\boxed{-x^2\sin x + 2x\cos x}$$

Ex. 4 If $\sin x + \sin y = 1$ find the derivative of y with respect to x.

Implicit Differentiation

$$\cos x + \cos y \frac{dy}{dx} = 0$$

$$\cos y \frac{dy}{dx} = -\cos x$$

$$\boxed{\frac{dy}{dx} = -\frac{\cos x}{\cos y}}$$

Ex. 5 Find the equation of the tangent line of the following function at the point where $x = \frac{\pi}{6}$

$$y = \frac{\sin x}{\cos 2x}$$

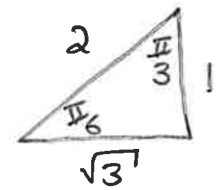
when $x = \frac{\pi}{6}$

$$y = \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{3}}$$

$$y' = \frac{\cos 2x(\cos x) - \sin x(-\sin 2x)(2)}{(\cos 2x)^2}$$

$$= \frac{\cos 2x \cos x + 2\sin x \sin 2x}{(\cos 2x)^2}$$

Recall Special Angle Δ's



$$y = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Point $\left(\frac{\pi}{6}, 1\right)$

$$\frac{dy}{dx} \text{ at } \frac{\pi}{6} \Rightarrow \frac{\cos \frac{\pi}{3} \cdot \cos \frac{\pi}{6} + 2\sin \frac{\pi}{6} \sin \frac{\pi}{3}}{\left(\cos \frac{\pi}{3}\right)^2}$$

$$= \frac{\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}}{\left(\frac{1}{2}\right)^2} \Rightarrow \frac{\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}}{\frac{1}{4}}$$

$$\rightarrow \frac{3\sqrt{3}}{4} \cdot \frac{4}{1} = \boxed{3\sqrt{3}} \leftarrow \text{that's slope.}$$

Now $y - y_1 = m(x - x_1)$

$$y - 1 = 3\sqrt{3}\left(x - \frac{\pi}{6}\right)$$

$$y = 3\sqrt{3}x - \frac{\sqrt{3}\pi}{2} + 1$$

$$\boxed{y = 3\sqrt{3}x + \frac{2 - \sqrt{3}\pi}{2}}$$

Homework Assignment

Practice Problems: 1acehjl, 2abdf,3-5