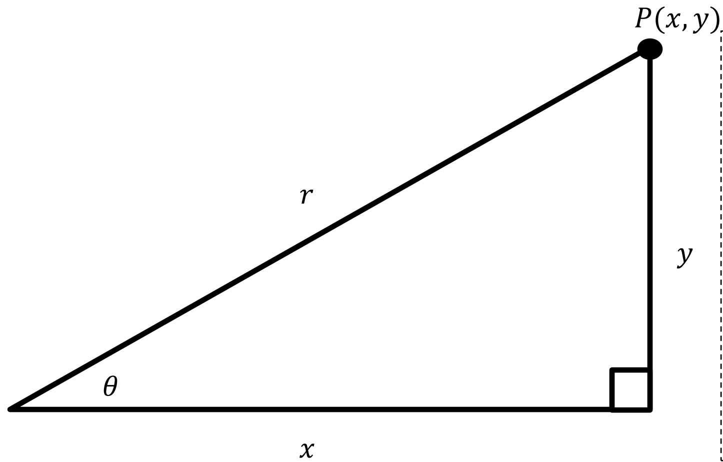
Section 7.1 – Trigonometric Identities

- We have often seen a number of scenarios where we solve equations – true for some values
- What we are going to look at here are called identities – true for all values of the variable

Consider a Right-Angle Triangle and the SOH CAH TOA ratios that we explored in earlier grades.



If point $P(x, y)$ is on the end of the terminal arm and it is in Standard Position, with a rotation of θ degrees or radians. Then from this we get our first six trigonometric ratios.

These are our first six trigonometric identities

Recall that: $r = \sqrt{x^2 + y^2}$

Trigonometric Ratios		
1. $\sin\theta = \frac{y}{r}$	2. $\cos\theta = \frac{x}{r}$	3. $\tan\theta = \frac{y}{x}$
4. $\csc\theta = \frac{r}{y}$	5. $\sec\theta = \frac{r}{x}$	6. $\cot\theta = \frac{x}{y}$

- The multiplicative principle gives us three more identities.
- Consider:

$$\sin\theta \cdot \csc\theta = \frac{y}{r} \cdot \frac{r}{y} = 1$$

$$\cos\theta \cdot \sec\theta = \frac{x}{r} \cdot \frac{r}{x} = 1$$

$$\tan\theta \cdot \cot\theta = \frac{y}{x} \cdot \frac{x}{y} = 1$$

This relationship implies that the given trigonometric functions are **reciprocals of one another**

The Reciprocal Identities		
1. $\csc\theta = \frac{1}{\sin\theta}$	2. $\sec\theta = \frac{1}{\cos\theta}$	3. $\cot\theta = \frac{1}{\tan\theta}$

- If we continue with the algebra and relationship of our original six ratios, we get a couple of other identities.

Using **sine and cosine functions** we see that:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r}\cdot\frac{r}{x} = \frac{y}{x} = \tan\theta$$

$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r}\cdot\frac{r}{y} = \frac{x}{y} = \cot\theta$$

The Quotient Identities	
4. $\frac{\sin\theta}{\cos\theta} = \tan\theta$	5. $\frac{\cos\theta}{\sin\theta} = \cot\theta$

- So what else is there?
- Well let's experiment with some algebra and equation manipulation
- An important minor detail to consider and understand is that:

$$(\sin\theta)^2 = \sin^2\theta$$

← This is the notation we use most often

That means that:

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{(y^2 + x^2)}{r^2}$$

But…

$$x^2 + y^2 = r^2$$

So, we get:

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{(y^2 + x^2)}{r^2} = \frac{r^2}{r^2} = 1$$

This gives us the identity:

$$\mathbf{\sin^2\theta + \cos^2\theta = 1}$$

- From the identity we just discerned, we can keep going in two different ways

Divide everything by $\sin^2\theta$	Divide everything by $\cos^2\theta$
$\sin^2\theta+\cos^2\theta=1$	$\sin^2\theta+\cos^2\theta=1$
$\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$	$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$
$1+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$	$\frac{\sin^2\theta}{\cos^2\theta}+1=\frac{1}{\cos^2\theta}$
$1+\left(\frac{\cos\theta}{\sin\theta}\right)^2=\frac{1}{\sin^2\theta}$	$\left(\frac{\sin\theta}{\cos\theta}\right)^2+1=\frac{1}{\cos^2\theta}$
$1+\cot^2\theta=\frac{1}{\sin^2\theta}$	$\tan^2\theta+1=\frac{1}{\cos^2\theta}$
$\boldsymbol{1+\cot^2\theta=\csc^2\theta}$	$\boldsymbol{\tan^2\theta+1=\sec^2\theta}$

The Pythagorean Identities		
6. $\sin^2\theta+\cos^2\theta=1$	7. $1+\cot^2\theta=\csc^2\theta$	8. $\tan^2\theta+1=\sec^2\theta$

This gives us 8 Basic Trigonometric Identities:

Basic Trigonometric Identities	
1. $\csc\theta=\frac{1}{\sin\theta}$	2. $\sec\theta=\frac{1}{\cos\theta}$
3. $\cot\theta=\frac{1}{\tan\theta}$	4. $\tan\theta=\frac{\sin\theta}{\cos\theta}$
5. $\cot\theta=\frac{\cos\theta}{\sin\theta}$	6. $\sin^2\theta+\cos^2\theta=1$
7. $1+\cot^2\theta=\csc^2\theta$	8. $\tan^2\theta+1=\sec^2\theta$

Strategies for Simplifying Trigonometric Expressions (Not Equations, There is No Equals Sign)

- We can use algebraic principles to demonstrate trigonometric simplification

1. Common Denominator

Algebra	Trigonometry
$a+\frac{b}{c}$	$\sin x+\frac{\sin x}{\cos x}$
$a\cdot\frac{c}{c}+\frac{b}{c}$	$\sin x\cdot\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}$
$\frac{ac}{c}+\frac{b}{c}$	$\frac{\sin x\cos x}{\cos x}+\frac{\sin x}{\cos x}$
$\frac{ac+b}{c}$	$\frac{\sin x\cos x+\sin x}{\cos x}$

2. Factoring (Particularly Difference of Squares)

Algebra	Trigonometry
$4-x^2$	$1-\cos^2 x$
$(2-x)(2+x)$	$(1-\cos x)(1+\cos x)$
Or	Or
x^2-z^2	$\sin^2 x-\cos^2 x$
$(x+z)(x-z)$	$(\sin x+\cos x)(\sin x-\cos x)$

3. Sine and Cosine Terms Only

a)	b)
$\frac{\sin x}{\csc x}+\frac{\cos x}{\sec x}$	$\frac{\tan x}{\sec x}$
$\frac{\sin x}{\frac{1}{\sin x}}+\frac{\cos x}{\frac{1}{\cos x}}$	$\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}$
$\sin x\cdot\sin x+\cos x\cdot\cos x$	$\frac{\sin x}{\cos x}\cdot\frac{\cos x}{1}$
$\sin^2 x+\cos^2 x$	$\sin x$
1	

4. **Multiplying by the Conjugate (Rationalizing the Denominator)**	
Algebra	Trigonometry
$\frac{\sqrt{3}}{\sqrt{2}+\sqrt{7}}$ $\frac{\sqrt{3}}{\sqrt{2}+\sqrt{7}} \cdot \frac{\sqrt{2}-\sqrt{7}}{\sqrt{2}-\sqrt{7}}$ $\frac{\sqrt{3}(\sqrt{2}-\sqrt{7})}{(\sqrt{2}+\sqrt{7})(\sqrt{2}-\sqrt{7})}$ $\frac{\sqrt{6}-\sqrt{21}}{2-7}$ $\frac{\sqrt{6}-\sqrt{21}}{-5}$ $-\frac{\sqrt{6}-\sqrt{21}}{5}$	$\frac{1}{1-\cos x}$ $\frac{1}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x}$ $\frac{1+\cos x}{(1-\cos x)(1+\cos x)}$ $\frac{1+\cos x}{1-\cos^2 x}$ If: $\sin^2 x + \cos^2 x = 1$ Then: $\sin^2 x = 1 - \cos^2 x$ $\frac{1+\cos x}{\sin^2 x}$

Example 1: Simplify

a) $(\sec^2 x - 1)\cot^2 x$

b) $\frac{2\cos x}{1-\sin^2 x}$

Solution 1:

a)

$(\sec^2 x - 1)\cot^2 x$

$\tan^2 x\,(\cot^2 x)$ ← Algebraic Manipulation of Identity #7

$\tan^2 x \cdot \frac{1}{\tan^2 x}$ ← Identity #3

1

b)

$\frac{2\cos x}{1-\sin^2 x}$

$\frac{2\cos x}{1 \;-\; \sin^2 x}$

$\frac{2\cos x}{\cos^2 x}$ ← Algebraic Manipulation of Identity #6

$\frac{2}{\cos x}$

$2\sec x$ ← Identity #2

Example 2: Simplify $\dfrac{\sin x}{1+\cos x}+\dfrac{\sin x}{1-\cos x}$

Solution 2: Before we can add them together, we need a Common Denominator (CD)

Multiply to get CD

$$\frac{\sin x}{1+\cos x}\cdot\frac{(1-\cos x)}{(1-\cos x)}+\frac{\sin x}{1-\cos x}\cdot\frac{(1+\cos x)}{(1+\cos x)}$$

Multiply Denominator →

$$\frac{\sin x\,(1-\cos x)}{1-\cos^2 x}+\frac{\sin x\,(1+\cos x)}{1-\cos^2 x}$$

Algebraic Manipulation of Identity #6 →

$$\frac{\sin x\,(1-\cos x)}{\sin^2 x}+\frac{\sin x\,(1+\cos x)}{\sin^2 x}$$

$$\frac{1-\cos x}{\sin x}+\frac{1+\cos x}{\sin x}$$

Cancel out $\sin x$ from numerator and denominator

$$\frac{2}{\sin x}$$

Identity #1

$$\mathbf{2\csc x}$$

Example 3: Simplify $\dfrac{1+\sin x}{\cos x}-\dfrac{\cos x}{1-\sin x}$

Solution 3: Before we can add them together, we need a Common Denominator

$$\frac{1+\sin x}{\cos x}\cdot\frac{(1-\sin x)}{(1-\sin x)}-\frac{\cos x}{1-\sin x}\cdot\frac{(\cos x)}{(\cos x)}$$

← Multiply to get CD

Multiply Numerators →

$$\frac{(1-\sin^2 x)}{\cos x\,(1-\sin x)}-\frac{\cos^2 x}{\cos x\,(1-\sin x)}$$

Factor out a Negative from the last two terms

Subtract Numerators →

$$\frac{1-\sin^2 x-\cos^2 x}{\cos x\,(1-\sin x)}\rightarrow\frac{1-(\sin^2 x+\cos^2 x)}{\cos x\,(1-\sin x)}$$

Identity #6 →

$$\frac{1-1}{\cos x\,(1-\sin x)}=\frac{0}{\cos x\,(1-\sin x)}=0$$

Subtract and Simplify

Example 4: Simplify $\dfrac{\sin x \cos x + \sin x}{\cos x + \cos^2 x}$

Solution 4: Look for Common Factors

$$\frac{\sin x\,(\cos x + 1)}{\cos x\,(1 + \cos x)}$$ ← Factor out Common Factor

$$\frac{\sin x\,(1 + \cos x)}{\cos x\,(1 + \cos x)}$$ ← Rearrange

$$\frac{\sin x\,(1 + \cos x)}{\cos x\,(1 + \cos x)}$$ ← Cancel Common Factor

Identity #4 → $$\frac{\sin x}{\cos x} = \tan x$$

Example 5: Determine the restrictions on: $\tan x + \sec x,\quad$ for $0 \leq x < 2\pi$

Solution 5: We have restrictions in Trigonometry Expressions, much like Algebraic Expressions, when the Denominator is 0.

$$\tan x + \csc x = \frac{\sin x}{\cos x} + \frac{1}{\sin x}$$

So, what matters is when is $\cos x = 0$ and when is $\sin x = 0$

$\cos x = 0$ when $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

$\sin x = 0$ when $x = 0\ and\ \pi$

$\tan x + \csc x$ **DOES NOT EXIST** at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Section 7.1 – Practice Problems

1. Match the given statement with a possible identity.

a) $\cot x$	
b) $\tan x$	
c) $\sec x$	
d) $\csc x$	
e) $\tan^2 x$	
f) $1+\tan^2 x$	
g) $\sin^2 x$	

A. $\dfrac{1}{\sin x}$	B. $\dfrac{1}{\cos x}$
C. $\dfrac{\sin x}{\cos x}$	D. $\dfrac{1}{\cos^2 x}$
E. $\dfrac{1}{\cot^2 x}$	F. $\dfrac{\cos x}{\sin x}$
G. $\dfrac{1}{\sin^2 x}$	H. $1-\cos^2 x$

2. Write with a Common Denominator, then simplify if possible.

a) $\dfrac{3}{2\sin x}-\dfrac{4}{\sin^2 x}$

b) $\dfrac{1}{1-\sin x}+\dfrac{1}{\sin x}$

c) $\dfrac{1+\frac{1}{\tan x}}{\frac{1}{\tan^2 x}}$

d) $\dfrac{1}{\sin^2 x} - 1$

e) $\sin x + \dfrac{\cos^2 x}{\sin x}$

f) $\dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x}$

g) $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}$

h) $\tan x-\dfrac{\sec^2 x}{\tan x}$

3. Factor, and then if possible, simply the expressions

a) $1-\sin^2 x$

b) $\sec^2 x-\tan^2 x$

c) $\tan^2 x-\tan^2 x\sin^2 x$

d) $\sec^2 x+\sec^2 x\tan^2 x$

e) $\sin^2 x \sec^2 x - \sin^2 x$

f) $\dfrac{\csc^2 x - 1}{\csc x - 1}$

g) $\cot^4 x + 2\cot^2 x + 1$

h) $1 - 2\sin^2 x + \sin^4 x$

i) $\sin^4 x - \cos^4 x$

j) $\sec^3 x - \sec^2 x - \sec x + 1$

4. Multiply, then simply using identities.

a) $(\sin x + \cos x)^2$

b) $\sin x\,(\csc x - \sin x)$

c) $(\csc x - 1)(\csc x + 1)$

d) $(2 - 2\cos x)(2 + 2\cos x)$

e) $(\csc x - \cot x)\,(\csc x + \cot x)$

f) $(\tan x + \sec x)\,(\tan x - \sec x)$

5. Rewrite the given expressions in terms of $\sin x$ only.

a) $\sin^2 x - \cos^2 x$

b) $\sec^2 x$

c) $\dfrac{\tan x + \sec x}{\cos x}$

d) $\dfrac{\sin x + \tan x}{1+\sec x}$

6. Rewrite the given expression in terms of $\cos x$ only

a) $\sin^2 x - \cos^2 x$

b) $(\sec x + 1)(\sec x - 1)$

c) $\sin x(\csc x - \sin x)$

d) $\dfrac{\cot x + \csc x}{\sin x}$

7. Rewrite in terms of Sine and Cosine only.

a) $\csc x + \cot x$

b) $\sec x + \tan x$

c) $\dfrac{1}{\tan x + \cot x}$

d) $\sec x - \dfrac{\cos x}{1 + \sin x}$

8. Determine all restrictions, $0 \le x < 2\pi$.

a) $\dfrac{\cot x}{1 + \sin x}$

b) $\dfrac{\sec x}{1 - \cos x}$

c) $\dfrac{1}{2\cos^2 x + \cos x - 1}$

d) $\cot x + \tan x$

9. Simplify the following trigonometric expressions.

a) $(\sec x \cdot \csc x - \cot x)(\sin x - \csc x)$

b) $\dfrac{\frac{\cot x + 1}{\cot x} - 1}{\frac{\cot x - 1}{\cot x} - 1}$

c) $\dfrac{\tan^2 x}{\cos^2 x + \sin^2 x + \tan^2 x}$

d) $\dfrac{\cos x \cdot \tan x + \sin x}{2 \tan x}$

e) $\dfrac{1-\sec^2 x}{\sec^2 x} - \cos^2 x$

f) $\dfrac{\sec x - \cos x}{\csc x - \sin x}$

g) $\dfrac{\cot x(\sin x + \tan x)}{\csc x + \cot x}$

h) $\dfrac{\sec x - \cos x}{\tan x}$

i) $\dfrac{\sec^2 x(1 + \csc x) - \tan x(\sec x + \tan x)}{\csc x(1 + \sin x)}$

j) $\dfrac{\csc^2 x + \sec^2 x}{\csc x \sec x}$

k) $\dfrac{\cos x + \cot x}{1 + \csc x}$

l) $\dfrac{\sec x}{\tan x + \cot x}$

See Website for Detailed Answer Key

Extra Work Space