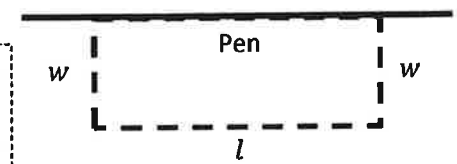
Section 6.5 – Applications of Quadratic Equations

- Some problems involve can maximizing or minimizing the quadratic function. Determining the vertex, and knowing if the vertex is a maximum or a minimum is the key for these (Ex 1-4)
- Some problems involve creating and solving a quadratic equation. Being able to solve a quadratic equation is the key for these (Ex 5-6)

Example 1:

A rectangular pen is to be built along the side of a barn. Find the maximum area that can be enclosed with $60m$ of fencing if the barn is one side of the enclosure.

Solution 1:



Let $w = width\ of\ the\ pen$, l = length of pen

$A = lw$

$2w + l = 60 \Rightarrow l = 60 - 2w$

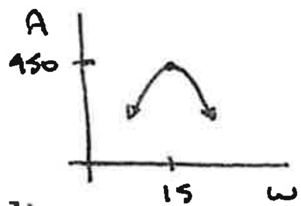
$A = (60 - 2w)w$

$A = 60w - 2w^2$

$A = -2w^2 + 60w$

$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$

$\left(-\frac{60}{2(-2)}, 0 - \frac{(60)^2}{4(-2)}\right) \rightarrow (15, 450)$



So the area is a maximum at the vertex.

Max Area $450\ m^2$

when $w = 15\ m$

$l = 60 - 2w$

$= 60 - 2(15)$

$= 30\ m$

Example 2:

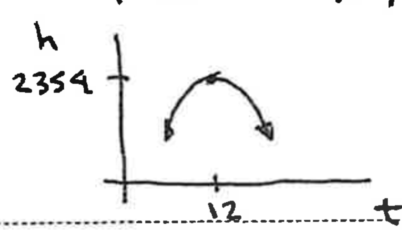
Mary stands on the top of a building and fires a gun upwards. The bullet travels according to the equation $h = -16t^2 + 384t + 50$, where h is the height of the bullet off the ground in feet at t seconds after it was fired.

a) How far is Mary above the ground when she fires the gun?
b) What is the bullet's maximum height above the ground?
c) How long does it take for the bullet to reach its greatest height?

Solution 2:

a) $t = 0$, $h = -16(0)^2 + 384(0) + 50 = 50$, 50 ft

b) $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right) \rightarrow \left(-\frac{384}{2(-16)}, 50 - \frac{(384)^2}{4(-16)}\right) \rightarrow (12, 2354)$



Max height $h = 2354$ ft

c) Max height when $t = 12$

Example 3:

Bob's Rent-a-Wreck rents 300 *cars at* $40 per day. For each $1 increase in cost of renting, 5 fewer cars are rented. For what rate should the cars be rented to produce the maximum income, and what is that income?

Solution 3:

- Let $R(x) = income\ from\ renting\ cars$
- If there is no change in rates, then $R(x) = 40 \cdot 300 = \$12\,000$ in income
- Let $x = increase\ in\ rate\ (in \$)$
- The new cost of renting a car is $(40 + x)$
- The number of cars rented is $(300 - 5x)$

Revenue = price per item × number of items

$R = (40 + x)(300 - 5x)$

$R = 12\,000 - 200x + 300x - 5x^2$

$R = -5x^2 + 100x + 12\,000$

$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right) \Rightarrow \left(-\frac{100}{2(-5)}, 12000 - \frac{100^2}{4(-5)}\right) \Rightarrow (10, 12\,500)$

max revenue \$12 500 when $x = 10$, so $40 + x$

$40 + 10$

\$50

Set price \$50

Example 4:

Find two numbers whose **difference** is 100 and the **sum of whose squares** is minimum.

Solution 4:

- Let $x = larger\ number$
- Let $y = smaller\ number$

$S = x^2 + y^2$

$x - y = 100 \Rightarrow y = x - 100$

$S = x^2 + y^2$

$S = x^2 + (x - 100)^2$

$S = x^2 + (x - 100)(x - 100)$

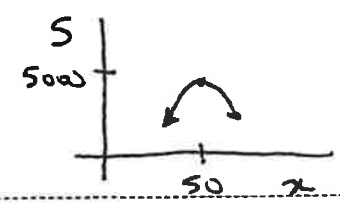
$S = x^2 + x^2 - 200x + 10000$

$S = 2x^2 - 200x + 10\,000$

$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$

$\left(-\frac{(-200)}{2(2)}, 10000 - \frac{(-200)^2}{4(2)}\right)$

$(50, 5000)$



$x = 50$

$y = x - 100$

$y = 50 - 100$

$y = -50$

The two numbers are : $\mathbf{50\ and - 50}$

The two numbers are 50, −50

Example 5:

The sum of a number and twice its reciprocal is $\frac{9}{2}$. Find the number.

Solution 5:

- Let $x = \textit{the number}$
- Then $\frac{1}{x}$ is *the reciprocal of a number*
- Then $\frac{2}{x}$ is *twice the reciprocal*

$$x + \frac{2}{x} = \frac{9}{2}$$

$$2x\left[x + \frac{2}{x} = \frac{9}{2}\right]$$

$$2x^2 + 4 = 9x$$

$$2x^2 - 9x + 4 = 0$$

(8; $-\frac{8}{2}$, $-\frac{1}{2}$; -9)

$$(x-4)(2x-1) = 0$$

$$x - 4 = 0 \text{ or } 2x - 1 = 0$$

$$x = 4 \text{ or } x = \frac{1}{2}$$

Check

$$4 + \frac{2}{4} \stackrel{?}{=} \frac{9}{2} \qquad \frac{1}{2} + \frac{2}{\frac{1}{2}} \stackrel{?}{=} \frac{9}{2}$$

$$\frac{8}{2} + \frac{1}{2} = \frac{9}{2} \qquad \frac{1}{2} + 4 = \frac{9}{2}$$

$$\frac{9}{2} = \frac{9}{2}$$

Soln: $x = 4$ and $x = \frac{1}{2}$

Example 6:

Ray and Ann ride a bicycle a distance of $4km$ each morning. They both finish at the same time but Ann starts 1 *minute* before Ray, and Ray travels $1km/h$ faster than Ann. What speeds are they travelling at?

Solution 6:

- Ann's time – Ray's time is $= 1\ min$ or $(1/60hr)$
- Remember: $Speed = \frac{Distance}{Time}$
- So: $Time = \frac{Distance}{Speed}$

	Speed (km/hr)	*Distance (km)*	*Time (hrs)*
Ann	x	4	$\frac{4}{x}$
Ray	$x+1$	4	$\frac{4}{x+1}$

$$\frac{4}{x} - \frac{4}{x+1} = \frac{1}{60}$$

$$60x(x+1)\left[\frac{4}{x} - \frac{4}{x+1} = \frac{1}{60}\right]$$

$$240(x+1) - 240x = x(x+1)$$

$$240x + 240 - 240x = x^2 + x$$

$$0 = x^2 + x - 240$$

$$x^2 + x - 240 = 0$$

$$x^2 + x - 240 = 0$$

(-240; -15, 16; 1)

$$(x-15)(x+16) = 0$$

$$x - 15 = 0 \quad x + 16 = 0$$

$$x = 15 \quad \sout{x = -16} \text{ R}$$

Ann: 15 km/h

Ray: 16 km/h

Section 6.5 – Practice Problems

1. The length of a rectangle is $4m$ more than the width. The area is $320m^2$. Find the length and the width.

2. Find two consecutive odd whole numbers such that the sum of their squares is 130.

3. Two planes travel at right angles to each other after leaving an airport at the same time; 1 hour later, they are $390km$ apart. If one plane travels $210km/h$ faster than the other, what is the speed of the slower plane?

4. The hypotenuse of a right triangle is $16cm$ long. One leg is $4cm$ longer than the other. Find the length of the legs.

5. The length and width of a rectangular sheet of plywood is $4ft\ by\ 8ft$. How much must be added equally to the length and width to double the area?

6. A boat takes 1 hour longer to go $36km$ up a river than to go down the river. If the boat travels $15km/hr$ in still water, what is the speed of the current?

Section 6.6 – Solving Systems of Non-Linear Equations

- When solving system of non-linear equations, we can similar strategies as linear
- The most straightforward way of doing this is to use the concept of equality

$$If \quad a = b \quad and \quad a = c \quad then \quad b = c$$

- We can use this concept to simplify the systems of equations by:
  - Writing both equations in **terms of one variable**
  - Setting them **equal to one another**
  - Solving for the **given variable**
  - **Substituting** back into the equation to solve for the remaining variable
  - Check our solution

Example: Solve the system: $y = x^2 - 3x - 4$ and $2x - y = 4$

Solution: Since one is already in terms of y, rearrange the other to also be in terms of y

$$2x - y = 4 \quad \rightarrow \quad y = 2x - 4$$

- Now since they are both equal to y we can set them equal to each other and solve for x

$y = x^2 - 3x - 4$ & $y = 2x - 4$

$x^2 - 3x - 4 = 2x - 4$

$x^2 - 5x = 0$

$x(x - 5) = 0$

$x = 0$ or $x - 5 = 0$

$x = 5$

Now, solve for y when $x = 0$

$y = 2x - 4$

$y = 2(0) - 4$

$y = -4$

And solve for y when $x = 5$

$y = 2x - 4$

$y = 2(5) - 4 = 6$

Soln: $(0, -4)$ & $(5, 6)$

Check: $(0, -4)$

$y = x^2 - 3x - 4$

$-4 = (0)^2 - 3(0) - 4$

$-4 = -4$

$2x - y = 4$

$2(0) - (-4) = 4$

$4 = 4$

$(5, 6)$

$y = x^2 - 3x - 4$

$6 = (5)^2 - 3(5) - 4$

$6 = 0$

$2x - y = 4$

$2(5) - 6 \stackrel{?}{=} 4$

$4 = 4$

System has Solutions:

$(0, -4)$ ***and*** $(5, 6)$

Example: Solve the system: $y = -\frac{1}{2}x^2 + 2x - 3$ and $y = x - 2$

Solution: Since they are both already in terms of y, set them equal to each other and solve for x

$$-2\left[-\frac{1}{2}x^2 + 2x - 3 = x - 2\right]$$

$$x^2 - 4x + 6 = -2x + 4$$

$$x^2 - 2x + 2 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4 - 8}}{2}$$

$$x = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \varnothing$$

no solutions

Graphs do not intersect

Example: Solve the system: $y = x^2 - 3x - 4$ and $2x - y = 3$

Solution: Since one is already in terms of y, rearrange the other to also be in terms of y

$$2x - y = 3$$

$$2x - 3 = y$$

$$y = 2x - 3$$

Now since they are both equal to y we can set them equal to each other and solve for x

$$x^2 - 3x - 4 = 2x - 3$$

$$x^2 - 5x - 1 = 0$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{25 - -4}}{2}$$

$$x = \frac{5 \pm \sqrt{29}}{2}$$

$$x = \frac{5 + \sqrt{29}}{2} \text{ \& } x = \frac{5 - \sqrt{29}}{2}$$

System has two solns

$$\left(\frac{5+\sqrt{29}}{2}, 2 + \sqrt{29}\right) \text{ \& } \left(\frac{5-\sqrt{29}}{2}, 2 - \sqrt{29}\right)$$

Solve for y when $x = \frac{5+\sqrt{29}}{2}$

$$y = 2x - 3$$

$$y = 2\left(\frac{5+\sqrt{29}}{2}\right) - 3$$

$$y = 5 + \sqrt{29} - 3$$

$$y = 2 + \sqrt{29}$$

Solve for y when $x = \frac{5-\sqrt{29}}{2}$

$$y = 2x - 3$$

$$y = 2\left(\frac{5-\sqrt{29}}{2}\right) - 3$$

$$y = 5 - \sqrt{29} - 3$$

$$y = 2 - \sqrt{29}$$

Example: Solve the system: $y = x^2 - x - 3$ and $y = 2x^2 - x + 7$

Solution: Since they are both already in terms of y, set them equal to each other and solve for x

$x^2 - x - 3 = 2x^2 - x + 7$

$0 = x^2 + 10$

$x^2 = -10$

$x = \pm\sqrt{-10}$

$x = \emptyset$

System has no solution

Graphs do not intersect.

Example: Solve the system: $x^2 - 4x + y + 1 = 0$ and $2x^2 - 2x - y + 2 = 0$

Solution:

$x^2 - 4x + y + 1 = 0$

$y = -x^2 + 4x - 1$

$2x^2 - 2x - y + 2 = 0$

$2x^2 - 2x + 2 = y$

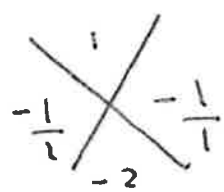
$-x^2 + 4x - 1 = 2x^2 - 2x + 2$

$0 = 3x^2 - 6x + 3$

$0 = 3(x^2 - 2x + 1)$

$0 = 3(x-1)(x-1)$

$x = 1$



Now, solve for y when $x = 1$

$y = -x^2 + 4x - 1$

$y = -(1)^2 + 4(1) - 1$

$y = 2$

Soln $(1, 2)$

Section 6.6 – Practice Problems

Find all the real solutions of the system of equations

1. $2x^2 - y = 1$ *and* $y = 5x + 2$

2. $x^2 - y = 3$ *and* $y = 3x + 7$

3. $x^2 = 2y$ *and* $y = x - \frac{1}{2}$

4. $x^2 + y = 4$ *and* $1 = 2x + y$

5. $3x^2 - 10y = 5$ *and* $x - y = -2$

6. $2x^2 - 3y = 2$ *and* $x - 2y = -2$

7. $x^2 + 2y = -2$ *and* $-2x + y = 1$

8. $x + y = 2$ *and* $y = 1 - x^2$

9. $y = x^2 - x$ *and* $y = 2x$

10. $y = x^2 - 6x$ *and* $y = x - 12$

11. $y = x^2 + 8x - 10$ *and* $y = 3x + 4$

12. $x^2 = y$ *and* $1 = 2x - y$

13. $x^2 + y = 9$ *and* $16 = 3x + 2y$

14. $x^2 - y = 10$ *and* $2x - 3y = -10$

15. $y = x^2$ *and* $x + y = 3$

16. $y + 2x^2 - 2 = 0$ *and* $3y - x - 3 = 0$

17. $y - x^2 = 0$ *and* $x^2 - 2x + y = 6$

18. $2x^2 + y = 9$ *and* $y - x^2 - 5x = 1$

19. Find all the points of intersection of the parabola $y = x^2 - 4x + 2$ and the $x - axis$

20. Find all the points of intersection of the parabola $y = 75x^2 - 33x + 157$ and the $y - axis$

Answer Key – Section 6.1

1. *Opens up*	2. *Opens down*
3. *Opens down*	4. *Opens down*
5. *Opens up*	6. *Opens down*
7. *See Website*	8. *See Website*
9. *See Website*	10. *See Website*
11. *See Website*	12. *See Website*
13. *See Website*	14. *See Website*
15. *See Website*	16. *See Website*
17. $y = -(x-2)^2$	18. $y = 2(x-1)^2 - 2$
19. $y = 3(x-1)^2 - 4$	20. $y = -\frac{3}{2}(x)^2 + 6$
21. $y = \frac{1}{2}(x+2)^2 + 1$	22. $y = 2(x-2)^2 - 2$

Answer Key – Section 6.2

1. $(1,3)$, *Minimum*	2. $(-2,11)$, *Maximum*
3. $(1,-2)$, *Minimum*	4. $\left(\frac{3}{2}, -\frac{7}{4}\right)$, *Maximum*
5. $(3,1)$, *Maximum*	6. $\left(\frac{5}{6}, -\frac{1}{12}\right)$, *Minimum*
7. *See Website*	8. *See Website*
9. *See Website*	10. *See Website*
11. *See Website*	12. *See Website*
13. *See Website*	14. *See Website*
15. *See Website*	16. *See Website*

Answer Key – Section 6.3

1. $x = 1; (x+1)(x-1); x = \emptyset$	2. $\frac{1}{2}; \pm\frac{1}{2};$ *No Roots*
3. *See Website*	4. $x = 0, x = 3$
5. $z = 4, z = -4$	6. $x = 2, x = -\frac{3}{2}$
7. $x = \frac{3}{2}, x = \frac{1}{3}$	8. $x = 2, x = -\frac{13}{6}$
9. $x = 0, x = \frac{8}{5}$	10. $y = \frac{5}{6}, y = -\frac{1}{2}$
11. $x = -5, x = -1$	12. $x = 3, x = -1$
13. $x = \frac{4}{5}, x = \frac{3}{2}$	14. $x = 4, x = 2$
15. $x = -3, x = \frac{1}{3}$	

Answer Key – Section 6.4

1. $x = 4, -4$	18. $0.54, -5.54$
2. $x = \sqrt{\frac{13}{3}}, -\sqrt{\frac{13}{3}}$	19. *No Solution*
3. $x = \sqrt{\frac{12}{5}}, -\sqrt{\frac{12}{5}}$	20. $1.54, -0.87$
4. $x = 1, -1$	21. $0.55, -1.45$
5. *No Solution*	22. $4.58, -0.58$
6. $x = -2, 6$	23. $5.37, -0.37$
7. $x = \frac{1 \pm \sqrt{12}}{2}$	24. $-1, 2$
8. $x = \frac{-2 \pm 3\sqrt{2}}{3}$	25. *No Solution*
9. $x = 1, -\frac{1}{5}$	26. $2.78, 0.72$
10. $x = 1, -\frac{1}{2}$	27. $1.87, -0.45$
11. $x = 1, -1$	28. $5.85, 0.68$
12. $x = 3, -1$	29. $\frac{17}{9}, \frac{13}{9}$
13. $3 \pm \sqrt{5}$	30. $-\frac{1}{2}$
14. $x = \frac{3 \pm \sqrt{7}}{2}$	31. $\frac{5}{2}$
15. $x = \frac{-2 \pm \sqrt{5}}{3}$	32. $\frac{2}{5}, \frac{3}{5}$
16. $x = \frac{-1 \pm \sqrt{5}}{2}$	33. $-2, 1$
17. $4.65, -0.65$	34. $1.19, -4.20$

Answer Key – Section 6.5

1. $w = 16, l = 20$
2. $7, 9$
3. $150 km/hr$
4. $9.14, 13.14$
5. $2.25 ft$
6. $3 km/hr$

Answer Key – Section 6.6

1.	$\left(-\frac{1}{2},-\frac{1}{2}\right),(3,17)$
2.	$(5,22),(-2,1)$
3.	$\left(1,\frac{1}{2}\right)$
4.	$(3,-5),(-1,3)$
5.	$(5,7),\left(-\frac{5}{3},\frac{1}{3}\right)$
6.	$(2,2),\left(-\frac{5}{4},\frac{3}{8}\right)$
7.	$(2,-3)$
8.	*No Solution*
9.	$(0,0),(3,6)$
10.	$(3,-9),(4,-8)$

11.	$(-7,-17),(2,10)$
12.	$(1,1)$
13.	$(2,5),\left(-\frac{1}{2},\frac{34}{4}\right)$
14.	$(4,6),\left(-\frac{10}{3},\frac{10}{9}\right)$
15.	$\left(\frac{-1+\sqrt{13}}{2},\frac{7-\sqrt{13}}{2}\right),\left(\frac{-1-\sqrt{13}}{2},\frac{7+\sqrt{13}}{2}\right)$
16.	$\left(\frac{-1+\sqrt{73}}{12},\frac{35+\sqrt{73}}{36}\right),\left(\frac{-1-\sqrt{73}}{12},\frac{35-\sqrt{73}}{36}\right)$
17.	$\left(\frac{1+\sqrt{13}}{2},\frac{7+\sqrt{13}}{2}\right),\left(\frac{1-\sqrt{13}}{2},\frac{7-\sqrt{13}}{2}\right)$
18.	$\left(-\frac{8}{3},-\frac{47}{9}\right),(1,7)$
19.	$(2+\sqrt{2},0),(2-\sqrt{2},0)$
20.	$(0,157)$