Section 6.3 – General and Special Angles

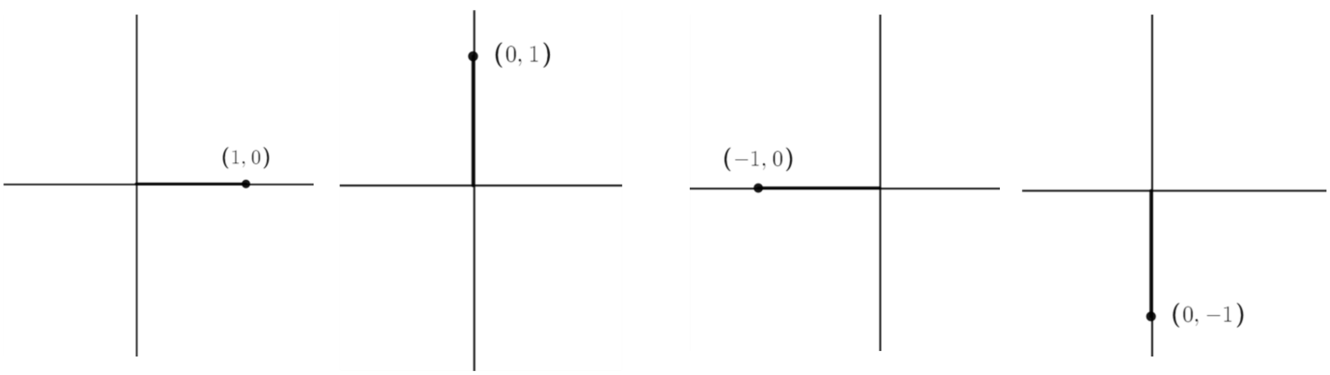
Quadrantal Angles

- Quadrantal angle are the easiest to calculate
- They are the angles where the **terminal arm is on** a the $\boldsymbol{x-axis\ or\ y-axis}$.

$$0° \leq \theta \leq 360°;\ \ 0 \leq x \leq 2\pi$$

- The easiest points to choose are $\mathbf{1\ unit}$ from the origin. **Where** $\boldsymbol{r = hyp = \sqrt{x^2+y^2} = 1}$

$\mathbf{0°\ or\ 0\ rad}$ $\qquad$ $\mathbf{90°\ or\ \frac{\pi}{2}}$ $\qquad$ $\mathbf{180°\ or\ \pi}$ $\qquad$ $\mathbf{270°\ or\ \frac{3\pi}{2}}$



<u>Example 1:</u> Evaluate each trigonometric function.

a) $\tan 90°$ $\qquad$ b) $\sin 180°$ $\qquad$ c) $\cos \pi$ $\qquad$ d) $\csc \frac{\pi}{2}$

<u>Solution 1:</u>

a) $\tan 90° = \frac{opposite}{adjacent} = \frac{y}{x} = \frac{1}{0} = Undefined$	b) $\sin 180° = \frac{opposite}{hypotenuse} = \frac{y}{r} = \frac{0}{1} = 0$
c) $\cos \pi = \frac{adjacent}{radius} = \frac{x}{r} = \frac{-1}{1} = -1$	d) $\csc \frac{\pi}{2} = \frac{radius}{opposite} = \frac{r}{y} = \frac{1}{-1} = -1$

This table helps to summarize the **Quadrantal Values of the Six trigonometric Functions**

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\mathbf{0°\ or\ 0}$	0	1	0	$Undefined$	1	$Undefined$
$\mathbf{90°\ or\ \frac{\pi}{2}}$	1	0	$Undefined$	1	$Undefined$	0
$\mathbf{180°\ or\ \pi}$	0	-1	0	$Undefined$	-1	$Undefined$
$\mathbf{270°\ or\ \frac{3\pi}{2}}$	-1	0	$Undefined$	-1	$Undefined$	0

- Recall that a co-terminal angle is greater or less than an angle between $\mathbf{0°}$ **and** $\mathbf{360°}$ or $\mathbf{0\ and\ 2\pi}$, by one or more full rotations in either a clockwise or counter-clockwise direction.

Example 2: Evaluate:

a) $\tan 1080°$

b) $\sec\frac{11\pi}{2}$

c) $\csc 7\pi$

Solution 2:

a) $\tan 1080°$

$$\tan 1080° = \tan 0°$$

$$\tan 0° = \frac{y}{x} = \frac{1}{0} = Undefined$$

b) $\sec\frac{11\pi}{2}$

$$\sec\frac{11\pi}{2} = \sec\frac{3\pi}{2}$$

$$\sec\frac{3\pi}{2} = \frac{r}{x} = \frac{1}{0} = Undefined$$

c) $\csc 7\pi$

$$\csc 7\pi = \csc\pi$$

$$\csc\pi = \frac{r}{y} = \frac{1}{0} = Undefined$$

Special Angles: $30°$ and $\frac{\pi}{6}$ $60°$ and $\frac{\pi}{3}$ $45°$ and $\frac{\pi}{4}$

- Two triangles in trigonometry are especially significant, we can calculate them exactly
- They are the $45° - 45° - 90°$ triangle and the $30° - 60° - 90°$ triangle.

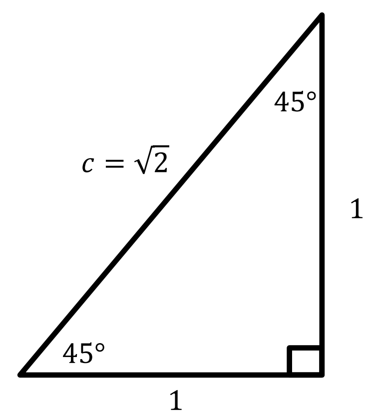
The $45° - 45° - 90°$ Triangle

- Since the triangle has two equal angles, it is an isosceles
- Since trigonometric function are based on ratios we can use any numbers, use 1 for simplicity
- By Pythagoras' Theorem:

$$c^2 = a^2 + b^2 = 1^2 + 1^2 \quad \rightarrow \quad c = \sqrt{2}$$

Therefore,

$$Sin\ 45° = \frac{1}{\sqrt{2}} \qquad Cos\ 45° = \frac{1}{\sqrt{2}} \qquad Tan\ 45° = \frac{1}{1} = 1$$



The $30° - 60° - 90°$ Triangle

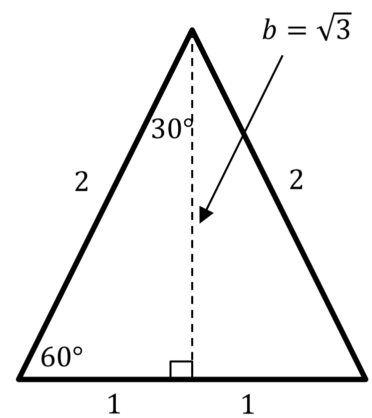
- Consider an equilateral triangle with all sides equal to 2
- Draw an altitude from a base to split the opposite 60 in half
- By Pythagoras' Theorem:

$$c^2 - a^2 = b^2 \quad \rightarrow \quad 2^2 - 1^2 = b^2 \quad \rightarrow \quad b = \sqrt{3}$$

Therefore,

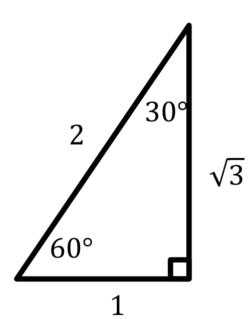
$$Sin\ 30° = \frac{1}{2} \qquad Cos\ 30° = \frac{\sqrt{3}}{2} \qquad Tan\ 30° = \frac{1}{\sqrt{3}}$$

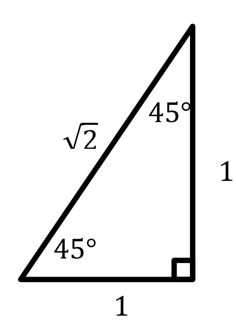
$$Sin\ 60° = \frac{\sqrt{3}}{2} \qquad Cos\ 60° = \frac{1}{2} \qquad Tan\ 60° = \sqrt{3}$$



Summary of Special Angles

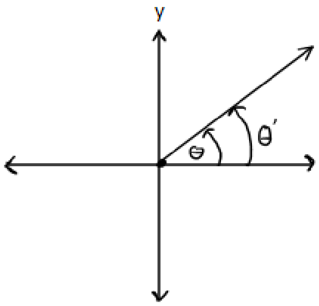
θ	$sin\ \theta$	$cos\ \theta$	$tan\ \theta$	$csc\ \theta$	$sec\ \theta$	$cot\ \theta$
$30°\ or\ \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$60°\ or\ \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
$45°\ or\ \frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1





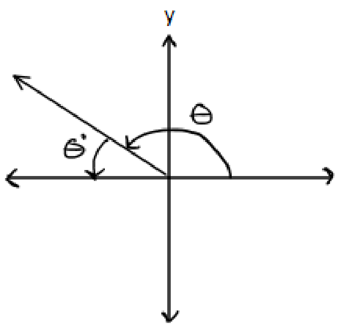
Reference Angles

- For and angle $\boldsymbol{\theta}$ **in Standard Position**, the **reference angle** is the **positive acute angle** $\boldsymbol{\theta'}$ that is **formed** with the **terminal side of** $\boldsymbol{\theta}$ **and the** $\boldsymbol{x-axis}$**.**
- Read that again…
- A reference angle is between $0°\ and\ 90°\ \left(0\ and\ \frac{\pi}{2}\right)$: $0° \le \theta' \le 90°$ or $0 \le \theta' \le \frac{\pi}{2}$



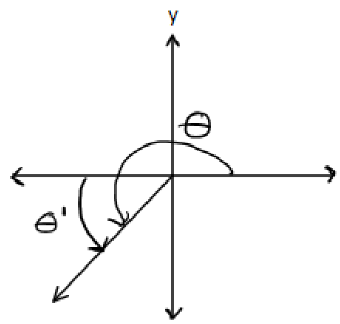
θ in Quadrant 1

$\theta' = \theta$



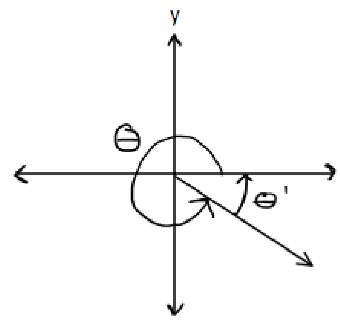
θ in Quadrant 2

$\theta' = 180° - \theta$



θ in Quadrant 3

$\theta' = \theta - 180°$

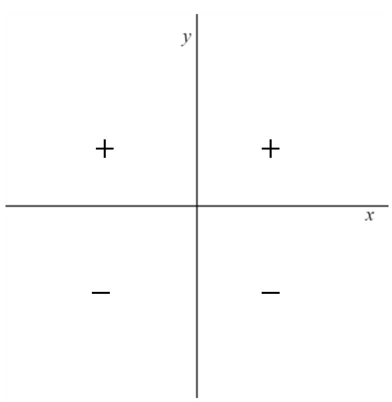


θ in Quadrant 4

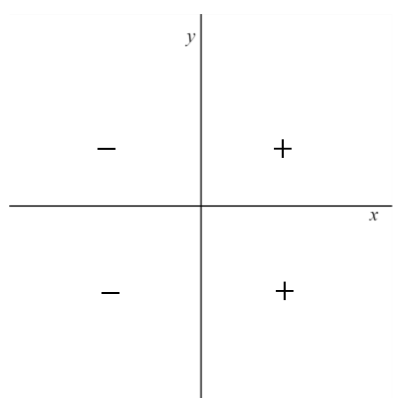
$\theta' = 360° - \theta$

<u>Recall that the Quadrant we find ourselves in, with respect to the reference angle, is important. It provides us with the positive/negative sign relationship of the ratio we are solving.</u>

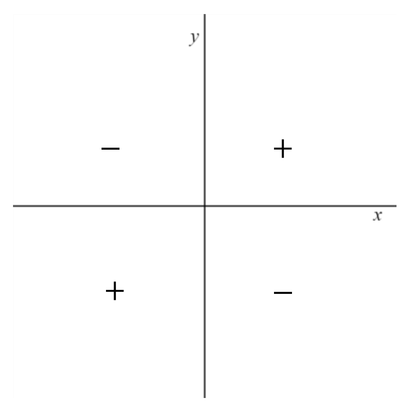
$\sin\theta \quad and \quad \csc\theta$



$\cos\theta \quad and \quad \sec\theta$



$\tan\theta \quad and \quad \cot\theta$



Example 3: Find the exact value of:

a) $\sin 300°$

b) $\tan\frac{5\pi}{6}$

c) $\csc -\frac{5\pi}{4}$

Solution 3: In each case we need to consider the reference angle.

a) $\sin 300°$ is in $Q4$.

Reference Angle:

$$360° - 300° = 60°$$

$\sin 60° = \frac{\sqrt{3}}{2}$ but in Q4, sin is negative

So,

$$\sin 300° = -\frac{\sqrt{3}}{2}$$

b) $\tan\frac{5\pi}{6}$ is in Q2

Reference Angle:

$$\pi - \frac{5\pi}{6} \rightarrow \frac{6\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{6}$$

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

But in Q2, tan is negative so,

$$\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$$

c) $\csc -\frac{5\pi}{4}$ is in Q2

Reference Angle:

$$-\frac{5\pi}{4} + 2\pi \rightarrow -\frac{5\pi}{4} + \frac{8\pi}{4} = \frac{3\pi}{4}$$

$$\pi - \frac{3\pi}{4} \rightarrow \frac{4\pi}{4} - \frac{3\pi}{4} = \frac{\pi}{4}$$

$$\csc\frac{\pi}{4} = \sqrt{2}$$

And in Q2, csc is positive so,

$$\csc\frac{\pi}{4} = \sqrt{2}$$

Finding θ. Quadrants Matter!

Example 4: Find the smallest possible θ in both degree and radian measure of:

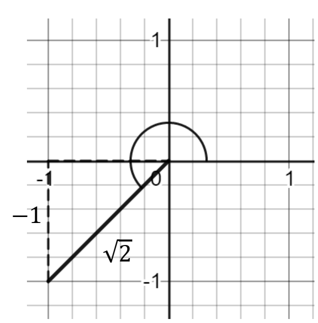
a) $\sin\theta = -\frac{1}{\sqrt{2}}$ b) $\cot\theta = -\frac{1}{\sqrt{3}}$

Solution 4:

a) First all, **where is Sine negativ**e? $Q3 \text{ and } Q4$. So, if we are finding the smallest, we are concerned with $Q3$.

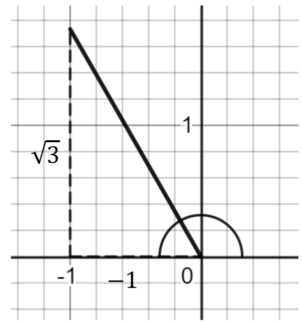
If we look at the ratio, we can deduce the reference angle that produces it is $45°$ or $\frac{\pi}{2}$

Therefore, in $Q3$, with a **reference angle of 45°**. We have:



$$\boldsymbol{\theta = 180° + 45° = 225°} \quad \boldsymbol{or} \quad \boldsymbol{\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}}$$

b) **Where is Cotangent negative**? Q2 and Q4. So, if we are finding the smallest, we are concerned with Q2.



If we look at the ratio, we can deduce the reference angle that produces it is $60° \ or \ \frac{\pi}{3}$

Therefore, in Q2, with a **reference angle of 60°**. We have:

$$\boldsymbol{\theta = 180° - 60° = 120° \quad or \quad \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}}$$

Example 5: Find all $\theta, 0° \leq \theta < 360°$ and $0 \leq \theta \leq 2\pi$ for which $\cos\theta = \frac{\sqrt{3}}{2}$

Solution 5:

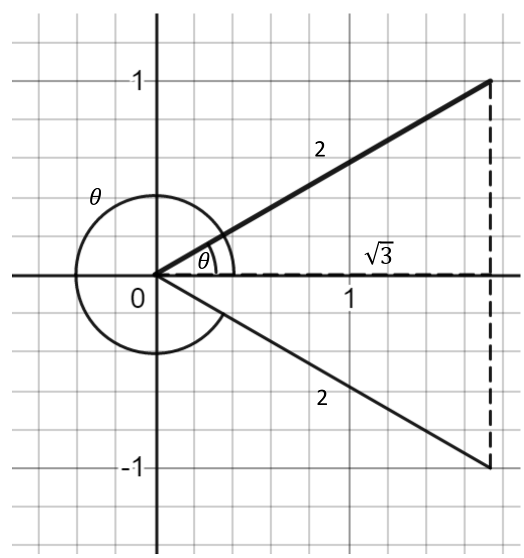
In this scenario, we are concerned with the entire grid and **where Cosine is Positive**. In our case, it is $\boldsymbol{Q1 \ and \ Q4}$. So, if we are finding all possible angels, we need to consider both scenarios.

If we look at the ratio, we can deduce the reference angle that produces it is $30° \ or \ \frac{\pi}{6}$

Therefore, in $Q1 \ and \ Q4$, we have a **reference angle of 30°**.

In $Q1$ we have:

$$\boldsymbol{\theta = 30° \quad or \quad \theta = \frac{\pi}{6}}$$



In $Q4$ we have:

$$\boldsymbol{\theta = 360° - 30° = 330° \quad or \quad \theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}}$$

Example 6: Find all $\theta, 0° \leq \theta < 360°$ and $0 \leq \theta \leq 2\pi$ for which $\csc\theta = -\sqrt{2}$

Solution 6:

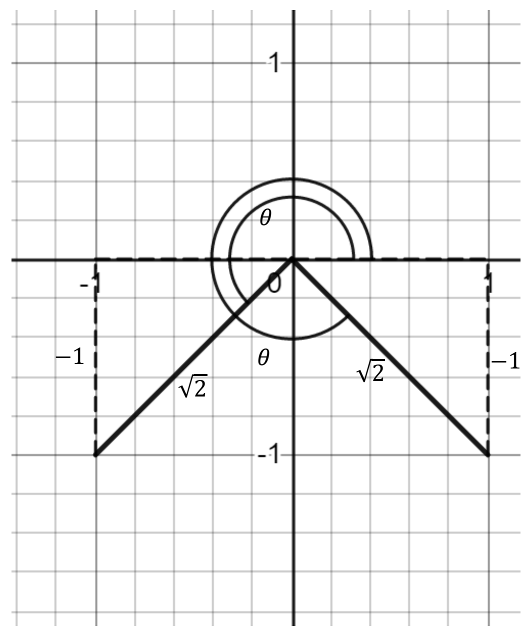
In this scenario, we are concerned with the entire grid and where Sine (because Cosecant is the inverse of Sine) is Negative. In our case, it is $Q3\ and\ Q4$. So, if we are finding all possible angels, we need to consider both scenarios.

If we look at the ratio, we can deduce the reference angle that produces it is $45°\ or\ \frac{\pi}{4}$

Therefore, in $Q3\ and\ Q4$, we have a **reference angle of** $\mathbf{45°}$ **or** $\boldsymbol{\frac{\pi}{4}}$.

In $Q3$ we have:

$$\boldsymbol{\theta = 180° + 45° = 225° \quad or \quad \theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}}$$



In $Q4$ we have:

$$\boldsymbol{\theta = 360° - 45° = 315° \quad or \quad \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}}$$

Get Comfortable with your Special Angle Ratios, Knowing them Saves Time and Tedium

Watch your Inverse Functions – Flip the Sine, Cosine, Tangent Ratios – For Cosecant, Secant, Cotangent

Section 6.3 – Practice Problems

1. Find the reference angle for each given angle in standard position

a) $150°$	b) $-150°$
c) $314°$	d) $-314°$
e) $612°$	f) $-537°$
g) $1100\ °$	h) $6325°$
i) $810°$	j) $-900°$
k) $\frac{7\pi}{6}$	l) $-\frac{21\pi}{4}$
m) $-\frac{19\pi}{5}$	n) $\frac{24\pi}{7}$
o) $\frac{17\pi}{3}$	p) $\frac{16\pi}{5}$

2. Determine the exact value of each trigonometric function, no calculator needed.

a) $\sin 120°$	b) $\cot 135°$
c) $\cos 330°$	d) $\tan 660°$
e) $\csc 1125°$	f) $\sec \frac{\pi}{6}$
g) $\sin \frac{5\pi}{4}$	h) $\tan \frac{11\pi}{6}$
i) $\csc \frac{19\pi}{6}$	j) $\cot \frac{13\pi}{3}$
k) $\cot(-240°)$	l) $\sec(-945°)$
m) $\cos(-\frac{5\pi}{3})$	n) $\tan(-\frac{29\pi}{6})$
o) $\sin(-\frac{20\pi}{3})$	p) $\csc(-\frac{27\pi}{4})$

3. For which value(s) of θ, $0° \leq \theta < 360°$ is each of the following trig functions undefined?

a) $\sin\theta$	b) $\cos\theta$
c) $\tan\theta$	d) $\cot\theta$
e) $\sec\theta$	f) $\csc\theta$

4. For which value(s) of θ, $0 \leq \theta < 2\pi$ is each of the following trig functions undefined?

a) $\sin\theta$	b) $\cos\theta$
c) $\tan\theta$	d) $\cot\theta$
e) $\sec\theta$	f) $\csc\theta$

5. Find the smallest positive θ in degrees (drawings help), for which:

a) $\sin\theta = -\frac{1}{2}$	b) $\tan\theta = -\sqrt{3}$

c) $\csc\theta = -\frac{2}{\sqrt{3}}$

d) $\sec\theta = -\sqrt{2}$

e) $\cot\theta = -\frac{1}{\sqrt{3}}$

f) $\cos\theta = -\frac{\sqrt{3}}{2}$

6. Find the smallest positive x in radians, for which:

a) $\sin x = -\dfrac{\sqrt{3}}{2}$

b) $\cot x = -\sqrt{3}$

c) $\csc x = -\sqrt{2}$

d) $\sec x = -\frac{2}{\sqrt{3}}$

e) $\tan x = -1$

f) $\cos x = -\frac{1}{2}$

7. Find the exact value of each expression, no calculator needed. Recall $sin^2\theta = (\sin\theta)^2$

a) $\sin 60°$

b) $2\sin 30° \cos 30°$

c) $\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{6}$

d) $\sin^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4}$

e) $\sec^2 60° - \tan^2 60°$

f) $\csc^2 \frac{\pi}{6} - \cot^2 \frac{\pi}{6}$

g) $2 \sin^2 \frac{\pi}{6}$

h) $1 - \cos \frac{\pi}{3}$

i) $\tan\frac{\pi}{3}$

j) $\dfrac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}}$

8. Find an angle x such that $x \neq y, 0 \leq x < 2\pi,$ and $\sin x = \sin y$

a) $y = \dfrac{\pi}{6}$

b) $y = \dfrac{7\pi}{4}$

c) $y = \dfrac{11\pi}{6}$

d) $y = \dfrac{4\pi}{3}$

9. Find an angle x such that $x \neq y, 0 \leq x < 2\pi,$ and $\cos x = \cos y$

a) $y = \frac{\pi}{6}$

b) $y = \frac{7\pi}{4}$

c) $y = \frac{7\pi}{6}$

d) $y = \frac{4\pi}{3}$

10. Find an angle x such that $x \neq y, 0 \leq x < 2\pi,$ and $\tan x = \tan y$

a) $y = \frac{\pi}{6}$

b) $y = \frac{7\pi}{4}$

c) $y = \frac{11\pi}{6}$

d) $y = \frac{4\pi}{3}$

11. Determine all possible values of x by special angles, $0 \leq x < 2\pi$ Drawings Help!

a) $\cos x = \frac{\sqrt{3}}{2}$

b) $\sin x = -\frac{1}{2}$

c) $\tan x = -1$

d) $\csc x = 2$

e) $\sec x = -\sqrt{2}$

f) $\sin x = -1$

g) $\cot x = undefined$

h) $\cos x = 0$

i) $\csc x = undefined$

j) $\sec x = -1$

k) $\cot x = -\frac{1}{\sqrt{3}}$

l) $\csc x = -\sqrt{2}$

12. Find the exact values of $\sin 3x$ and $\sin\left(\frac{x}{3}\right)$ for the given values of x.

a) $x = 0$

b) $x = \frac{\pi}{2}$

c) $x = -\frac{\pi}{2}$

d) $x = -\pi$

13. Find the exact values of $\cos 3x$ and $\cos\left(\frac{x}{3}\right)$ for the given values of x.

a) $x = 0$

b) $x = \frac{\pi}{2}$

c) $x = -\frac{\pi}{2}$

d) $x = -\pi$

14. Choose a variety of angles for $\sin\theta$ and $\sin(-\theta)$. How and when do the value between the two differ or stay the same

15. Choose a variety of angles for $\cos\theta$ and $\cos(-\theta)$. How and when do the value between the two differ or stay the same

See Website for Detailed Answer Key

Extra Work Space