Section 6.2 – Trigonometric Functions on a Cartesian Grid

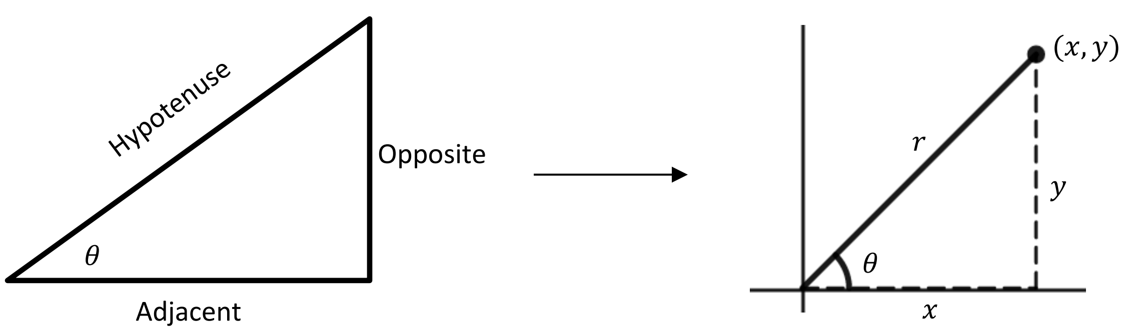
- Were back to looking at **Acute Angles with the $x-axis$ (reference angles)** and the relationship they give our triangles created as our **Terminal Arm rotates** around in **Standard Position**
- In Grade 10 and 11 we talked all about the **Three Trigonometric Ratios**
- In this grade, we include **three more** and discuss our **measure in terms of Radians**

Trigonometric Ratios of Acute Angles

For a given acute angle θ with have these ratios

Seen These Before	These are New, but Not Too Different
$\boldsymbol{Sine}$: $\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{y}{r}$	$\boldsymbol{Cosecant}$: $\csc\theta = \frac{Hypotenuse}{Opposite} = \frac{r}{y}$
$\boldsymbol{Cosine}$: $\text{Cos}\,\theta = \frac{Adjacent}{Hypotenuse} = \frac{x}{r}$	$\boldsymbol{Secant}$: $\sec\theta = \frac{Hypotenuse}{Adjacent} = \frac{r}{x}$
$\boldsymbol{Tangent}$: $\text{Tan}\,\theta = \frac{Opposite}{Adjacent} = \frac{y}{x}$	$\boldsymbol{Cotangent}$: $\text{Cot}\,\theta = \frac{Adjacent}{Opposite} = \frac{x}{y}$

But on a Cartesian Grid



This is our good old

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Way of considering a right-angle triangle.

Remember, the Quadrant the Terminal Arm is in determines:

When x or y is positive/negative

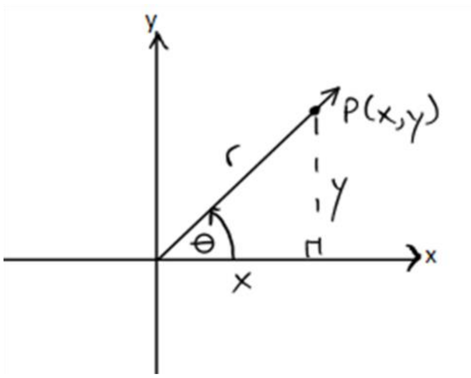
r is always positive, it is the radius

$$r^2 = \sqrt{x^2 + y^2}$$

Algebraic Sings of the Trigonometric Functions

- We spent a lot of time on these in Grade 11, but here it is again
- Also, we are looking at the **Three New Trigonometric Ratios:** $\sec\theta, \csc\theta, \cot\theta$
- It is important to notice that they are the RECIPROCAL of the first three
- This relationship means the Quadrant and Sign scenario doesn't change.

Quadrant 1



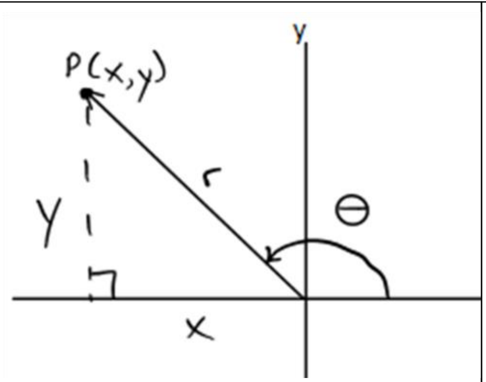
For θ in $Q1$: $x > 0, y > 0$

$\sin\theta = \frac{y}{r} = \frac{+}{+} = +$ $\qquad$ $\csc\theta = \frac{r}{y} = \frac{+}{+} = +$

$\cos\theta = \frac{x}{r} = \frac{+}{+} = +$ $\qquad$ $\sec\theta = \frac{r}{x} = \frac{+}{+} = +$

$\tan\theta = \frac{y}{x} = \frac{+}{+} = +$ $\qquad$ $\cot\theta = \frac{x}{y} = \frac{+}{+} = +$

Quadrant 2



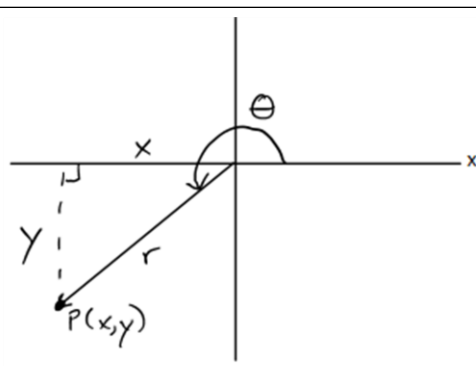
For θ in $Q2$: $x < 0, y > 0$

$\sin\theta = \frac{y}{r} = \frac{+}{+} = +$ $\qquad$ $\csc\theta = \frac{r}{y} = \frac{+}{+} = +$

$\cos\theta = \frac{x}{r} = \frac{-}{+} = -$ $\qquad$ $\sec\theta = \frac{r}{x} = \frac{+}{-} = -$

$\tan\theta = \frac{y}{x} = \frac{+}{-} = -$ $\qquad$ $\cot\theta = \frac{x}{y} = \frac{-}{+} = -$

Quadrant 3



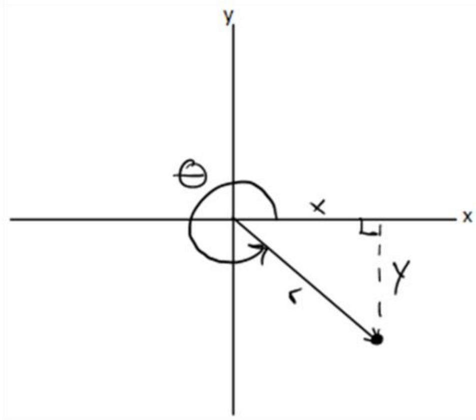
For θ in $Q3$: $x < 0, y < 0$

$\sin\theta = \frac{y}{r} = \frac{-}{+} = -$ $\qquad$ $\csc\theta = \frac{r}{y} = \frac{+}{-} = -$

$\cos\theta = \frac{x}{r} = \frac{-}{+} = -$ $\qquad$ $\sec\theta = \frac{r}{x} = \frac{+}{-} = -$

$\tan\theta = \frac{y}{x} = \frac{-}{-} = +$ $\qquad$ $\cot\theta = \frac{x}{y} = \frac{-}{-} = +$

Quadrant 4



For θ in $Q4$: $x > 0, y < 0$

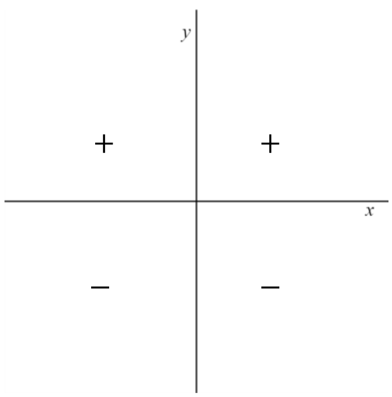
$$\sin\theta = \frac{y}{r} = \frac{-}{+} = - \qquad \csc\theta = \frac{r}{y} = \frac{+}{-} = -$$

$$\cos\theta = \frac{x}{r} = \frac{+}{+} = + \qquad \sec\theta = \frac{r}{x} = \frac{+}{+} = +$$

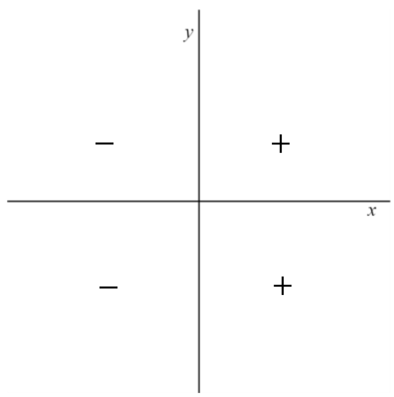
$$\tan\theta = \frac{y}{x} = \frac{-}{+} = - \qquad \cot\theta = \frac{x}{y} = \frac{+}{-} = -$$

Summary

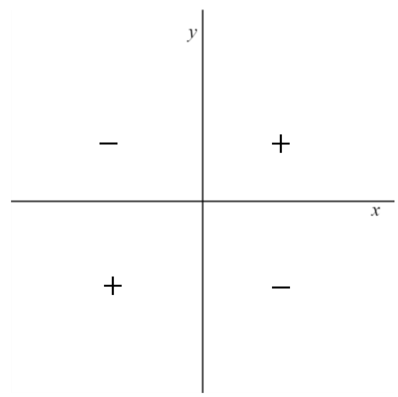
$\sin\theta$ *and* $\csc\theta$



$\cos\theta$ *and* $\sec\theta$



$\tan\theta$ *and* $\cot\theta$



<u>Example 1:</u> What Quadrant has $\boldsymbol{\sin\theta < 0, \tan\theta > 0}$

<u>Solution 1:</u>

Where is $\sin\theta < 0$? $Q3$ *and* $Q4$

Where is $\tan\theta > 0$? $Q1$ *and* $Q3$

So, we need the Quadrant where we have overlap. In this case: $\boldsymbol{Q3}$

Example 2: Determine $\cos\theta, when \csc\theta = -\frac{7}{\sqrt{3}}$ and $\tan\theta < 0$

Solution 2: Draw a picture to see what is going on.

$\csc\theta < 0$ in Q3 and Q4, and $\tan\theta < 0$ in Q2 and Q4, so we need to make our sketch in Q4.

We know that $\boldsymbol{r = 7}$ (can't be negative)

and

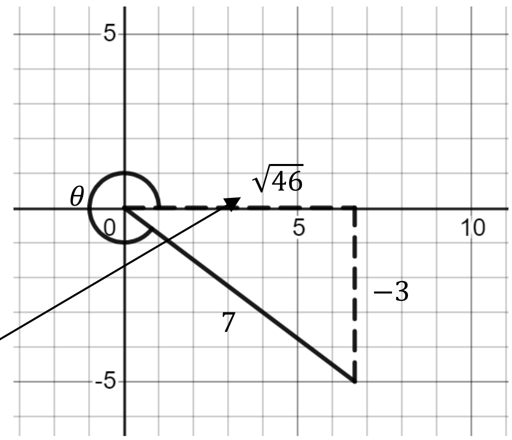
$\boldsymbol{y = -\sqrt{3}}$, so, we use Pythagorean Theorem for x

$$x^2 = (7)^2 - \left(-\sqrt{3}\right)^2$$

$$x^2 = 49 - 3 = 46$$

$$x = \sqrt{46}$$

Since cosine is positive in Q4, $x = \sqrt{46}$



Therefore:

$$\boldsymbol{\cos\theta = \frac{\sqrt{46}}{7}}$$

Example 3: Determine $\sec\theta, when \cot\theta = \frac{2}{5}$

Solution 3: Draw a picture to see what is going on.

$\sec\theta = \frac{r}{x}$ and we have $\cot\theta = \frac{x}{y}$ so we need to solve for r be aware $\cot\theta > 0$ in Q1 and Q3

We know $x = 2$ and $y = 5$ so we can use Pythagorean Theorem for r

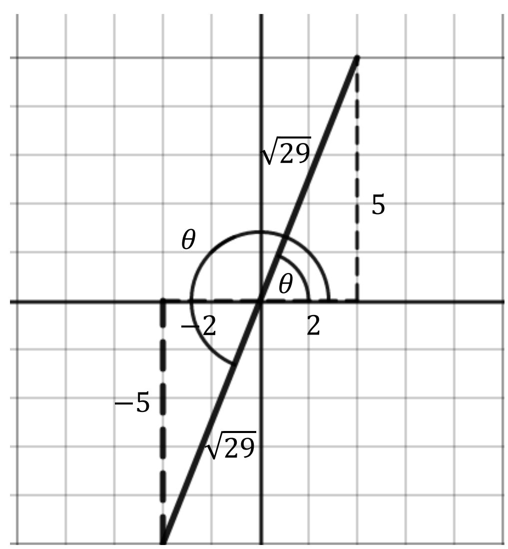
$$r^2 = 2^2 + 5^2$$

$$r^2 = 4 + 25 = 29$$

$$r = \sqrt{29}$$

Therefore:

$$\boldsymbol{\sec\theta = -\frac{\sqrt{29}}{2}} \textit{ and } \boldsymbol{\sec\theta = \frac{\sqrt{29}}{2}}$$



Example 4: Determine $\cot\theta, when \sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta < 0$

Solution 4: Draw a picture to see what is going on.

$\cos\theta < 0$ in Q2 and Q3, and $\sin\theta > 0$ in Q1 and Q2, so we need to make our sketch in Q2.

We know that $\boldsymbol{r = 2}$ (can't be negative)

and

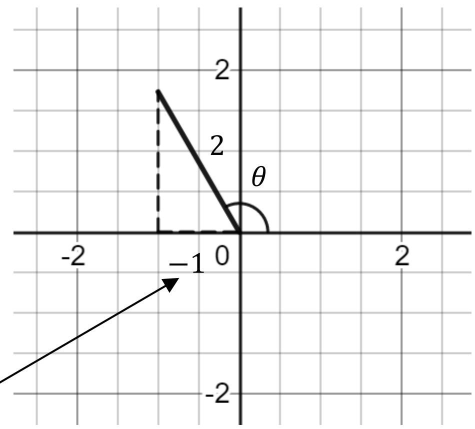
$\boldsymbol{y = \sqrt{3}}$, so, we use Pythagorean Theorem for x

$$x^2 = (2)^2 - \left(\sqrt{3}\right)^2$$

$$x^2 = 4 - 3 = 1$$

$$x = 1$$

Since cosine is negative in Q2, $\boldsymbol{x = -1}$



Therefore:

$$\boldsymbol{\cot\theta = -\frac{1}{\sqrt{3}}}$$

Example 5: Given the point $(-2, 1)$ on the terminal side of angle θ, what are the six trigonometric ratios?

Solution 5: Draw a picture to see what is going on. $(-2, 1)$ is located in Q2.

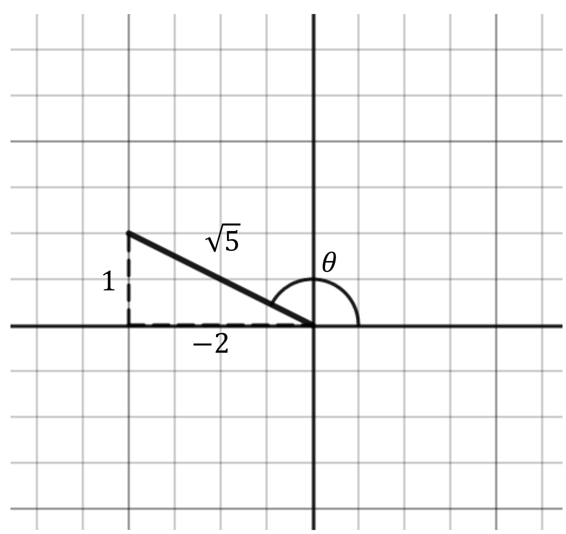
$\boldsymbol{\sin\theta = \frac{1}{\sqrt{5}}}$ $\boldsymbol{\csc\theta = \sqrt{5}}$

$\boldsymbol{\cos\theta = -\frac{2}{\sqrt{5}}}$ $\boldsymbol{\sec\theta = -\frac{\sqrt{5}}{2}}$

$\boldsymbol{\tan\theta = -\frac{1}{2}}$ $\boldsymbol{\cot\theta = -2}$

$$\boldsymbol{x = -2}, \quad \boldsymbol{y = 1}$$

$$r^2 = (-2)^2 + 1^2 = 5 \quad \rightarrow \quad \boldsymbol{r = \sqrt{5}}$$



Example 6: Determine $\sin\theta$ *and* $\cos\theta$, *if* θ is in Standard Position of a terminal arm in the position: $2x - 5y = 0,\ x \le 0$

Solution 6: Draw a picture to see what is going on. $2x - 5y = 0 \quad \rightarrow \quad y = \frac{2}{5}x$

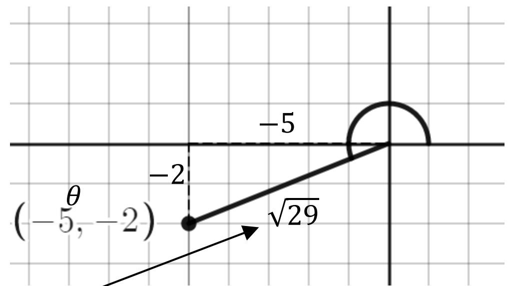
Since $x \le 0, x = -5$, so $y = -2$, in order to give a slope of: $\frac{2}{5}$ in Q3

Use Pythagorean Theorem for r

$$r^2 = (-5)^2 + (-2)^2$$

$$r^2 = 25 + 4 = 29$$

$$r = \sqrt{29}$$



$$\mathbf{\cos\theta = -\frac{5}{\sqrt{29}} \quad \textit{and}\ \sin\theta = -\frac{2}{\sqrt{29}}}$$

Example 7: Determine the coordinate 12 units from the origin in Q4 and $\tan\theta = -\frac{3}{4}$

Solution 7: Draw a picture to see what is going on.

Since tangent is negative in Q4, we know that:

$y = -3\ and\ x = 4$

Use Pythagorean Theorem for r

$$r^2 = (3)^2 + (-4)^2$$

$$r^2 = 9 + 16 = 25$$

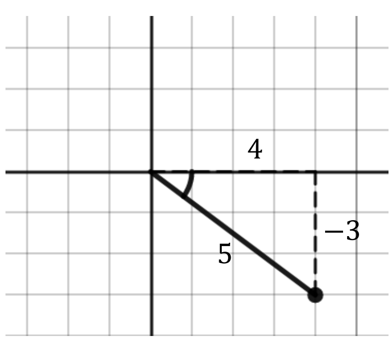
$$r = \sqrt{25} = 5$$

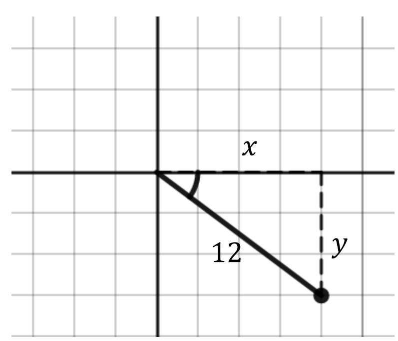
We solve this by proportion, consider the two graphs below. Compare the ratios of sides using similarity.

$$\frac{x}{12} = \frac{4}{5} \quad \rightarrow \quad x = \frac{48}{5}$$

$$\frac{y}{12} = \frac{-3}{5} \quad \rightarrow \quad y = -\frac{36}{5}$$

The coordinate is: $\left(\mathbf{-\frac{36}{5}, \frac{48}{5}}\right)$





Example 8: If $\sin\theta = \frac{\sqrt{3}}{2}$ find:

a) $\csc\theta$ b) $\cos(90° - \theta)$

Solution 8:

a) Since $\sin\theta$ *and* $\csc\theta$ are reciprocals of each other: $\boldsymbol{\csc\theta = \frac{2}{\sqrt{3}}}$

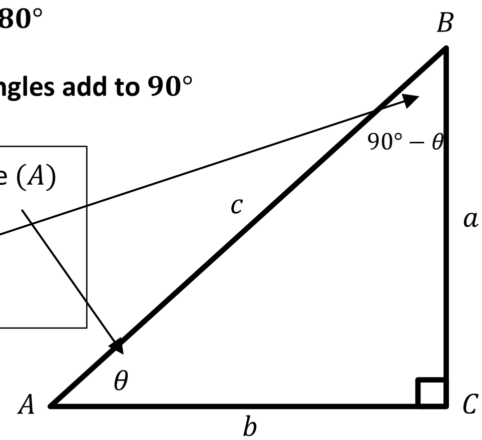
b) For this question consider the angle relationship in a right-angle triangle

Angles in a triangle add to 180°

In a **right-angle triangle**, the **other two angles add to 90°**

Which means that if the bottom angle (A) is θ, the top angle (B) must be:

$\boldsymbol{90° - \theta}$



And if we consider SOH CAH TOA:

$$\sin\theta = \frac{a}{c} \quad and \quad \cos(90° - \theta) = \frac{a}{c}$$

Therefore: $\sin\theta = \cos(90° - \theta)$ so $\boldsymbol{\cos(90° - \theta) = \frac{\sqrt{3}}{2}}$

Note:

This comes back around for Inverse Trigonometric Functions in Calculus

$\sin\theta = \cos(90° - \theta)$ $\cos\theta = \sin(90° - \theta)$ $\tan\theta = \cot(90° - \theta)$

$\cot\theta = \tan(90° - \theta)$ $\sec\theta = \csc(90° - \theta)$ $\csc\theta = \sec(90° - \theta)$

Section 6.2 – Practice Problems

1. Find the missing value of the right-angle triangle with sides a, b and hypotenuse c

a) $a = 5, b = 12, c = ?$

b) $a = 2, b = 3, c = ?$

c) $a = 15, c = 17, b = ?$

d) $b = 2\sqrt{2}, c = 3, a = ?$

e) $c = 3\sqrt{5}, b = 6, a = ?$

f) $c = \sqrt{17}, a = 2\sqrt{2}, b = ?$

2. Determine the Quadrant in which θ is found, given the following information.

a) $\sin\theta > 0, \quad \sec\theta > 0$

b) $\tan\theta < 0, \quad \cos\theta > 0$

c) $\csc\theta > 0, \quad \cot\theta < 0$	d) $\cos\theta < 0, \quad \csc\theta < 0$
e) $\sin\theta < 0, \quad \tan\theta < 0$	f) $\cot\theta > 0, \quad \sec\theta < 0$
g) $\tan\theta < 0, \quad \csc\theta > 0$	h) $\cos\theta > 0, \quad \sec\theta < 0$
i) $\sin\theta < 0, \quad \cot\theta < 0$	j) $\tan\theta < 0, \quad \sec\theta > 0$

3. Find the value of the indicated function

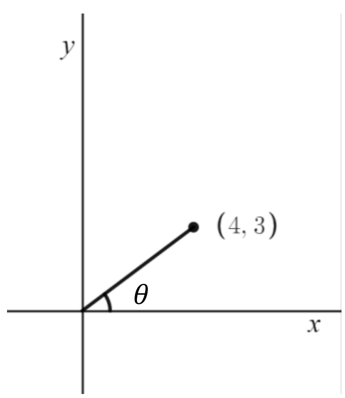
a) If $\csc\theta = 2,\ \sin\theta = ?$	b) If $\cos\theta = -\frac{2}{3},\ \sec\theta = ?$
c) If $\tan\theta = -5,\ \cot\theta = ?$	d) If $\sin\theta = -0.23,\ \csc\theta = ?$
e) If $\sec\theta = 2.35,\ \cos\theta = ?$	f) If $\cot\theta = -2.4,\ \tan\theta = ?$

4. Find the acute angle θ, given the following information for the trigonometric functions

a) $\sin 30° = \cos\theta$ so $\theta = ?$	b) $\tan 65° = \cot\theta$ so $\theta = ?$
c) $\sec 25° = \csc\theta$ so $\theta = ?$	d) $\cos\frac{\pi}{4} = \sin\theta$ so $\theta = ?$
e) $\cot\frac{\pi}{6} = \tan\theta$ so $\theta = ?$	f) $\csc\frac{\pi}{3} = \sec\theta$ so $\theta = ?$

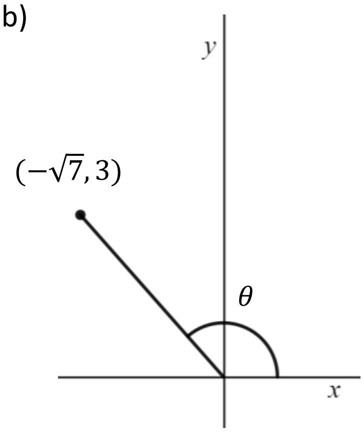
5. Given the point on the Terminal Arm in Standard Position, Evaluate all six trigonometric functions

a)



$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

b)



$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

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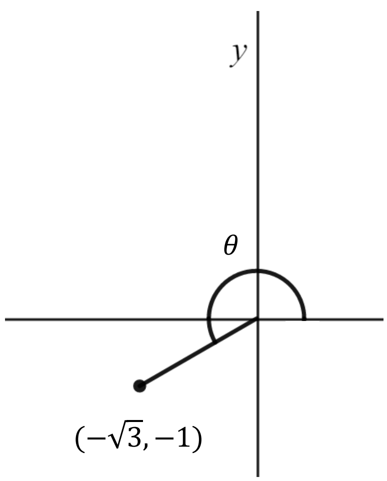
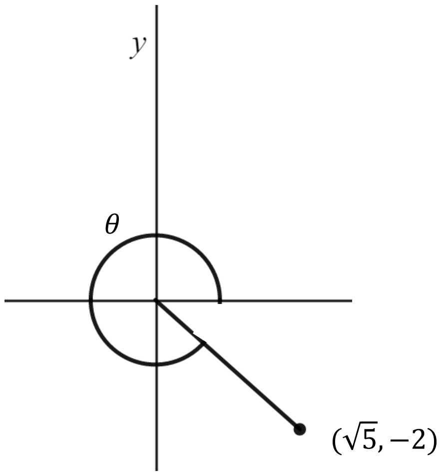
c)



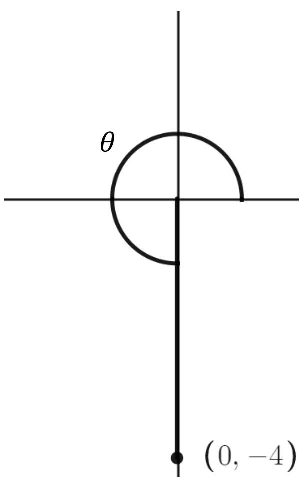
$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

d)



$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

e)



$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

6. Given the one trigonometric function, find the other 5.

a) $\sin\theta = \frac{5}{13}$ θ *is in* $Q1$

$\sin\theta = \frac{5}{13}$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

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b) $\tan\theta = \frac{8}{15}$ θ *is in* $Q3$

$\sin\theta =$	$\cos\theta =$	$\tan\theta = \frac{8}{15}$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

c) $\sec\theta = \frac{3}{2}$ $\theta\ is\ in\ Q4$

$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta = \frac{3}{2}$	$\cot\theta =$

d) $\csc\theta = 3$ $\tan\theta < 0$

$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta = 3$	$\sec\theta =$	$\cot\theta =$

e) $\cot\theta = -2.4 \quad \sin\theta > 0$

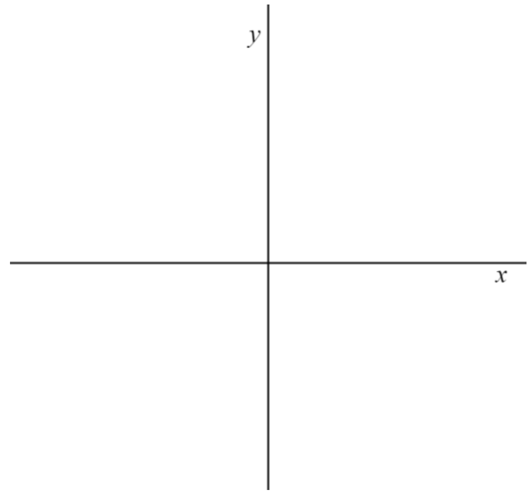
$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta = -2.4$

f) $\cos\theta = -0.238 \quad \tan\theta > 0$

$\sin\theta =$	$\cos\theta = -0.238$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

7. Find the six trigonometric functions of θ if θ is an angle created by the Terminal Arm in Standard Position and is located on the cartesian plane according to the given function.

a) $3x + 5y = 0, \;\; x \geq 0$



$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

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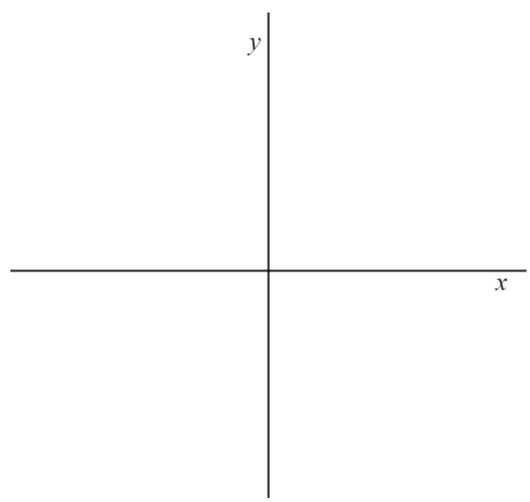
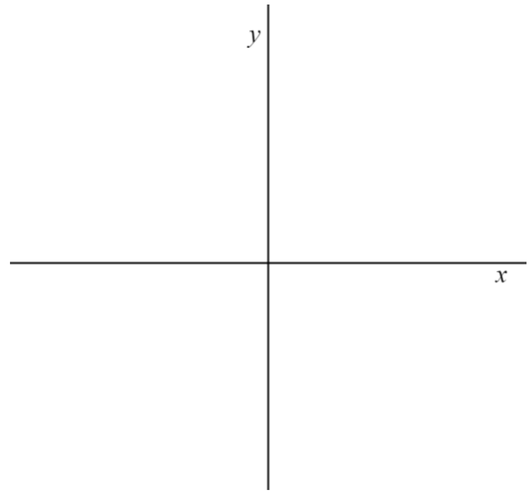
b) $2x - 3y = 0, \;\; y \leq 0$



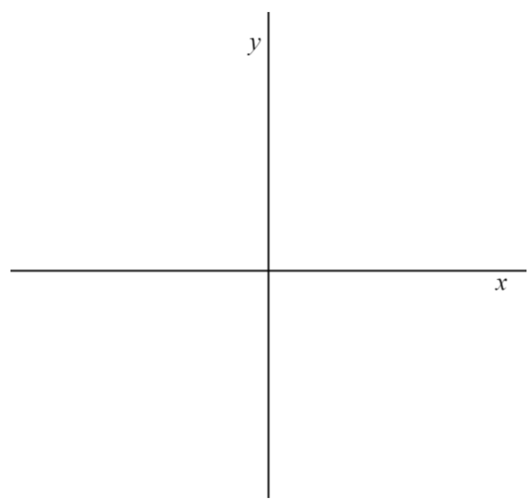
$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

c) $\sqrt{5}x + 2y = 0, \quad y \leq 0$



$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

d) $x = 0 \quad y \leq 0$



$\sin\theta =$	$\cos\theta =$	$\tan\theta =$
$\csc\theta =$	$\sec\theta =$	$\cot\theta =$

8. Determine the coordinates of the point at the given distance from the origin in the stated quadrant, if θ is its position angle.

a) Distance of 10, $Q2$, $\sin\theta = \frac{3}{5}$

b) Distance of 3, $Q3$, $\tan\theta = 1$

c) Distance of 8, $Q1$, $\sec\theta = 2$

d) Distance of 8, $Q2$, $\csc\theta = \frac{13}{5}$

9. Let B be an acute angle where $\sin B = a$. Find $\csc B$ and $\cos(90° - B)$ in terms of a.

10. Let P be an acute angle where $\cos P = b$. Find $\sec P$ and $\sin(\frac{\pi}{2} - P)$ in terms of b

11. The terminal side of angle θ in Standard Position, goes through the intersection point of the given curves. Find the intersection point, then find $\sin\theta$ *and* $\cos\theta$

a) $2x - y = 10$

$3x + y = 5$

$\sin\theta =$

$\cos\theta =$

b) $y = x^2 + 4x$

$y = -4x - 16$

$\sin\theta =$

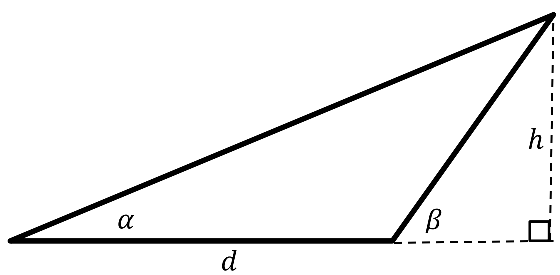
$\cos\theta =$

12. Find all angles of θ, $0 \leq \theta < 360°$, where $\sin\theta = cos\theta$

13. If $1 + \sin\theta = 3\sin\theta$, *where* $\tan\theta < 0$. Find $\cos\theta$.

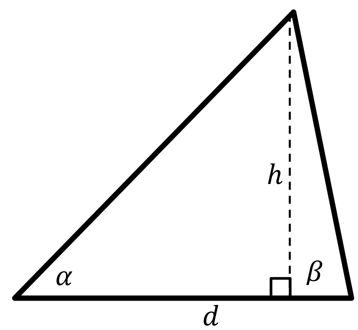
14. Show that:

$$h = \frac{d}{\cot\alpha - \cot\beta}$$



15. Show that:

$$h = \frac{d}{\cot\alpha + \cot\beta}$$



See Website for Detailed Answer Key

Extra Work Space