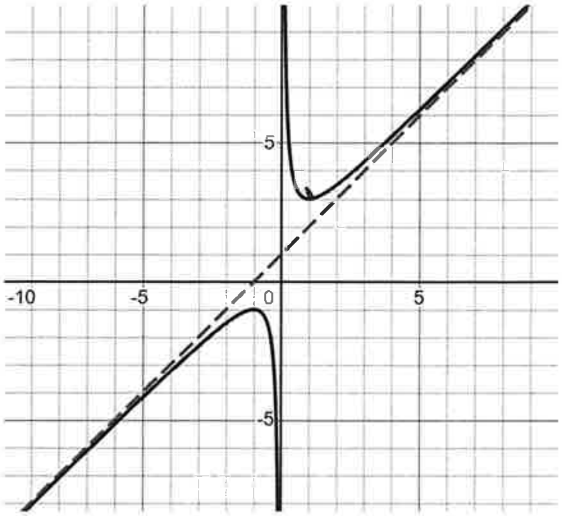
5.6 Slant Asymptotes

Consider the function

$$y = x + 1 + \frac{1}{x}$$

For large values of x, $1/x$ is small and so the values of $f(x)$ are close to $x + 1$. This means that the graph of f is close to the graph of the line $y = x + 1$. This line is called a ***slant asymptote*** or an ***oblique asymptote*** of the graph of f.



With respect to limits, we have the following

$$\begin{aligned}\lim_{x \to \infty} [f(x) - (x+1)] &= \lim_{x \to \infty} \left[\left(x + 1 + \frac{1}{x}\right) - (x+1)\right] \\ &= \lim_{x \to \infty} \frac{1}{x} \\ &= 0\end{aligned}$$

In general, **the line $y = mx + b$ is a slant asymptote** if the **vertical distance** between the curve $y = f(x)$ and the line, **approaches 0 as x gets infinitely large** (positive of negative).

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0 \qquad \Big| \qquad \lim_{x \to -\infty} [f(x) - (mx + b)] = 0$$

For rational functions, **slant asymptotes** occur when the **degree of the numerator** is **one more** than the **degree of the denominator**. Consider the previous example.

$$f(x) = x + 1 + \frac{1}{x}$$

It can be written…

$$f(x) = \frac{x^2 + x + 1}{x}$$

The equation of a slant asymptote can be found by long division

Ex. 1 Find the slant asymptote of the following function

$$y = \frac{2x^3 - 3x^2 + x - 3}{x^2 + 1}$$

$$\begin{array}{r|l} & 2x - 3 \\ \hline x^2+1 & 2x^3 - 3x^2 + x - 3 \\ & 2x^3 \qquad +2x \\ \hline & -3x^2 - x - 3 \\ & -3x \qquad -3 \\ \hline & -x \end{array}$$

$$f(x) = \frac{2x^3 - 3x^2 + x - 3}{x^2 + 1} = 2x - 3 + \frac{(-x)}{x^2 + 1}$$

$$f(x) = 2x - 3 - \frac{x}{x^2 + 1}$$

$$\lim_{x \to \infty} \left[f(x) - (2x - 3) \right]$$

$$\left[2x - 3 - \frac{x}{x^2 + 1} - 2x + 3 \right]$$

$$\lim_{x \to \infty} \left[-\frac{x}{x^2 + 1} \right] \rightarrow \lim_{x \to \infty} \frac{-\frac{x}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}} = \lim_{x \to \infty} \frac{-\frac{1}{x}}{1 + \frac{1}{x^2}}$$

$$\lim_{x \to \infty} \frac{-\frac{1}{\infty}}{1 + \frac{1}{\infty}} = \frac{0}{1 + 0} = 0$$

Therefore the line $2x - 3$ is a slant asymptote

Ex. 2 Find the slant asymptote of the following function. Then sketch the curve and identify the:

a) Domain
b) Intercepts
c) Symmetry
d) Asymptotes
e) Intervals of Increase and Decrease
f) Local Minimum and Maximum Values
g) Concavity and Points of Inflection

$-(x^2-x-1)$

$\frac{1\pm\sqrt{1-4(1)(-1)}}{2}, \frac{1\pm\sqrt{5}}{2}$

$$y=\frac{(1+x-x^2)}{x-1}$$

A: Domain

$\{x \mid x\in\mathbb{R}; x\neq 1\}$

B: y-int: $(0,-1)$

x-int: $\left(\frac{1+\sqrt{5}}{2},0\right)$

$\left(\frac{1-\sqrt{5}}{2},0\right)$

C: $f(-x)=\frac{1-x-x^2}{-x-1}$

= no symmetry

$$x-1\,\overline{\big)\,-x^2+x+1}\quad -x$$
$$\underline{-x^2+x}$$
$$1$$

$\rightarrow f(x) = -x+\frac{1}{x-1}$

$\lim\limits_{x\to\infty} f(x)-(-x) \rightarrow -x+\frac{1}{x-1}+x$

$=\frac{1}{x-1} \rightarrow \lim\limits_{x\to\infty}\frac{1}{\infty}=\boxed{0}$

D: VA: $x=1$

HA: none since we have a slant asymptote

SA: $y=-x$

E: Use $f(x)=-x+\frac{1}{x-1}$

$f'(x)=-1-\frac{1}{(x-1)^2}$

↑ always positive

ppc $x=1$

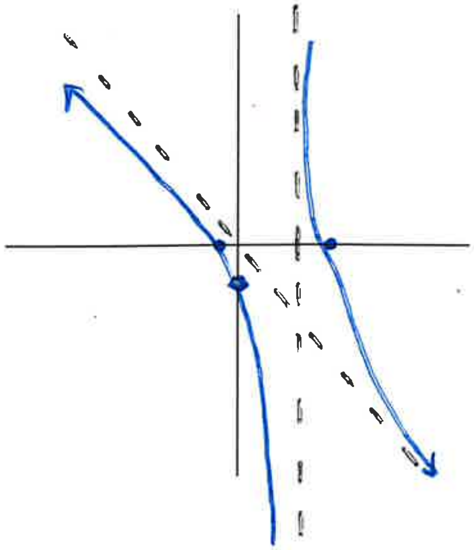
Interval
$(-\infty,1)$
$(1,\infty)$
Always negative
Always decreasing

F: None

G: $\frac{2}{(x-1)^3}$

Interval	$(x-1)^3$	
$(-\infty,1)$	$-$	CD
$(1,\infty)$	$+$	CU

no inflection



Homework Questions

Practice Problems: #1-3