5.5 A Procedure for Curve Sketching

Plotting points and graph the curve is simply not sufficient. As we have seen in the previous sections we have a lot of things to consider. Our goal is to get the graphical representation as accurate as possible and calculus is useful in choosing specific domains.

The procedure we use for sketching a curve $y = f(x)$ is to assemble the information as follows.

A. Domain.

The first step is to find the Domain of the function

B. Intercepts.

Next, we find the x −intercept(s) (if any) and the y −intercept.

C. Symmetry.

If $f(-x) = f(x)$, then f is even and symmetric about the $x-$ axis. If $f(-x) = -f(x)$, then f is odd and symmetric about the origin.

D. Asymptotes

Vertical Asymptotes: The vertical asymptotes are found by equating the denominator to 0 and dividing out any common factors (these create holes). If $x = a$ is a vertical asymptote then the limits

$$\lim_{x \to a^-} f(x) \qquad \lim_{x \to a^+} f(x)$$

Are identified as either ∞ or $-\infty$.

Horizontal Asymptotes: The line $y = L$ is a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \lim_{x \to -\infty} f(x) = L$$

E. Intervals of Increase and Decrease

We calculate $f'(x)$ and use the First Derivative Test.

F. Local Maximum and Minimum Values

We find the critical numbers of f and use the First Derivative Test (or Second Derivative Test).

G. Concavity and Points of Inflection

We calculate $f''(x)$ and use the Test for Concavity. Inflection occurs where the concavity changes.

H. Sketch the Curve. **Put it all together!**

Ex. 1 Discuss the curve below and include the headings A-H.

$$y = 3x^5 - 5x^3$$

A: Domain: All Real #'s

B: $y = x^3(3x^2 - 5)$ → $0 = x^3(3x^2-5)$ x-ints

$x = 0$ (Passes through) $x = \pm\sqrt{\frac{5}{3}} \doteq \pm 1.3$

$(0,0)$ $(1.3, 0)$ $(-1.3, 0)$

y-int: $(0,0)$

C: $f(-x) = 3(-x)^5 - 5(-x)^3$

$= -3x^5 + 5x^3$

$= -(3x^5 - 5x^3)$

so

$f(-x) = -f(x)$ ← odd function Symmetric about the origin

D: No vertical asymptotes

$\lim_{x\to\infty} x^3(3x^2-5) = \infty$

$\lim_{x\to-\infty} x^3(3x^2-5) = -\infty$

E: $y' = 15x^4 - 15x^2$

$0 = (15x^2)(x^2-1)$ crit pts: $0, \pm 1$

Interval	$15x^2$	(x^2-1)	$f'(x)$	$f(x)$
$(-\infty, -1)$	+	+	+	increasing
$(-1, 0)$	+	−	−	decreasing
$(0, 1)$	+	−	−	decreasing
$(1, \infty)$	+	+	+	increasing

$f(-1) = 2$ local max

$f(1) = -2$ local min

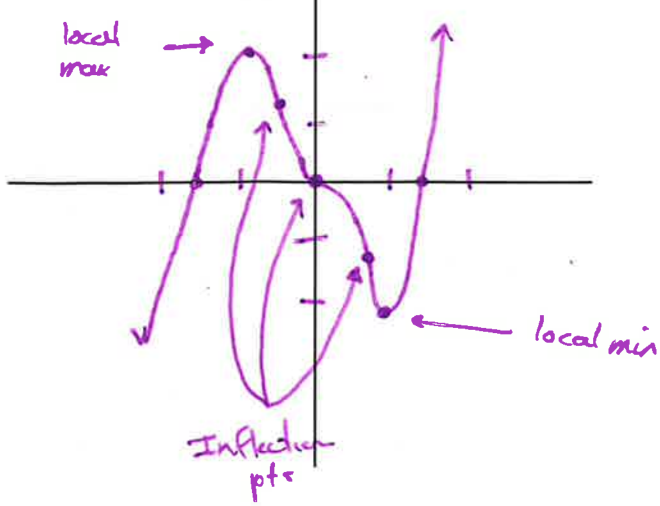
F: $y''(x) = 60x^3 - 30x$

$= 30x(2x^2-1)$

ct pt's $x = 0$

$x = \pm\frac{1}{\sqrt{2}}$

Interval	$30x$	$(2x^2-1)$	$f''(x)$	$f(x)$
$(-\infty, -\frac{1}{\sqrt{2}})$	−	+	−	CD
$(-\frac{1}{\sqrt{2}}, 0)$	−	−	+	CU
$(0, \frac{1}{\sqrt{2}})$	+	−	−	CD
$(\frac{1}{\sqrt{2}}, \infty)$	+	+	+	CU



G: $f''(x) = 0$ when $x = 0, \pm\frac{1}{\sqrt{2}}$

Inflection pts

$f(0) = 0$ $f(\frac{1}{\sqrt{2}}) \doteq -1.2$

$f(-\frac{1}{\sqrt{2}}) \doteq 1.2$

Ex. 2 Discuss the curve below and include the headings A-H.

$$y = \frac{1}{1+x^2}$$

A: Domain is all Real #'s

B: y-int: $(0,1)$

x-int: none

C: $f(-x) = \frac{1}{1+(-x)^2} = \frac{1}{1+x^2}$

$f(-x) = f(x)$ so even

Symmetric about y-axis

D: No vertical asymptote.

$$\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2}+\frac{x^2}{x^2}} \rightarrow \frac{0}{1} = 0$$

HA: 0

E: $f(x) = (1+x^2)^{-1}$

$$f'(x) = \frac{-1}{(1+x^2)^2}\cdot 2x = \frac{-2x}{(1+x^2)^2}$$

crit pt: $x = 0$

Interval	$-2x$	$(1+x^2)^2$	$f'(x)$	$f(x)$
$(-\infty,0)$	+	+	+	inc
$(0,\infty)$	−	+	−	dec

F: local max at $x=0$, $y=1$ $(0,1)$

G: $f'(x) = \frac{-2x}{(1+x^2)^2}$

$$f''(x) = \frac{(1+x^2)^2(-2) - (-2x)(2)(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4}$$

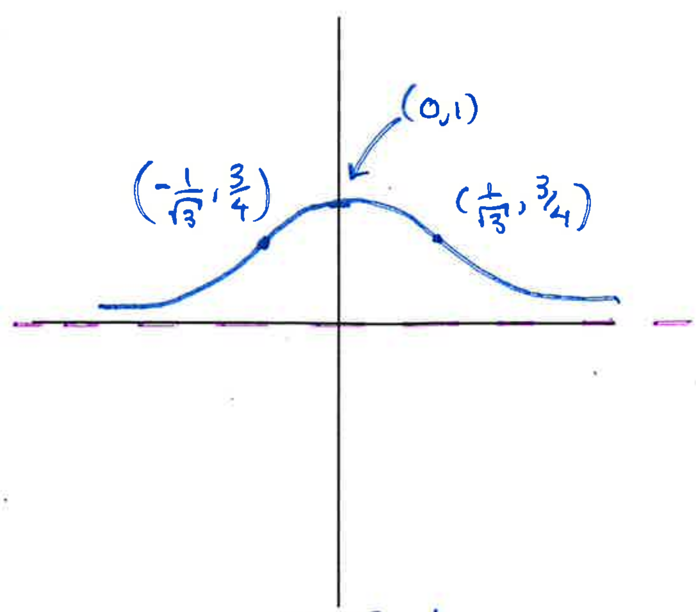
$$= \frac{-2(1+x^2)+8x^2}{(1+x^2)^3} \rightarrow \frac{-2-2x^2+8x^2}{(1+x^2)^3}$$

$$= \frac{6x^2-2}{(1+x^2)^3} \rightarrow \frac{2(3x^2-1)}{(1+x^2)^3}$$

always positive

crit pts: $\pm\frac{1}{\sqrt{3}}$

Interval	$(3x^2-1)$	$f''(x)$	$f(x)$
$(-\infty,-\frac{1}{\sqrt{3}})$	+	+	cu
$(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$	−	−	cd
$(\frac{1}{\sqrt{3}},\infty)$	+	+	cu



Points of Inflection

$(-\frac{1}{\sqrt{3}}, \frac{3}{4})$ and $(\frac{1}{\sqrt{3}}, \frac{3}{4})$

Ex. 3 Discuss the curve below and include the headings A-H.

$$y = \frac{x^2}{1-x^2}$$

A: Domain $\{x \mid x \in \mathbb{R};\ x \neq \pm 1\}$

B: y-int: $(0,0)$

x-int: $(0,0)$

C: Symmetry

$$f(-x) = \frac{(-x)^2}{1-(-x)^2} = f(x)$$

Symmetric about y-axis

D: VA: $x = \pm 1$

HA: $y = -1$

E: $f'(x) = \dfrac{(1-x^2)(2x) - x^2(-2x)}{(1-x^2)^2}$

$$= \frac{2x - 2x^3 + 2x^3}{(1-x^2)^2}$$

$$= \frac{2x}{(1-x^2)^2}$$

crit pts $x = 0$

ppc $x = \pm 1$

E: $f'(x)$ Interval

Interval	$2x$	$(1-x^2)^2$	$f'(x)$	$f(x)$
$(-\infty,-1)$	$-$	$+$	$-$	dec
$(-1,0)$	$-$	$+$	$-$	dec
$(0,1)$	$+$	$+$	$+$	inc
$(1,\infty)$	$+$	$+$	$+$	inc

F: $f(0) = 0$ ← local min

G: $f'(x) = \dfrac{2x}{(1-x^2)^2}$

$$f''(x) = \frac{(1-x^2)^2(2) - 2x(2)(1-x^2)(-2x)}{(1-x^2)^4}$$

$$= \frac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4}$$

$$= \frac{2(1-x^2) + 8x^2}{(1-x^2)^3} \rightarrow \frac{2 - 2x^2 + 8x^2}{(1-x^2)^3}$$

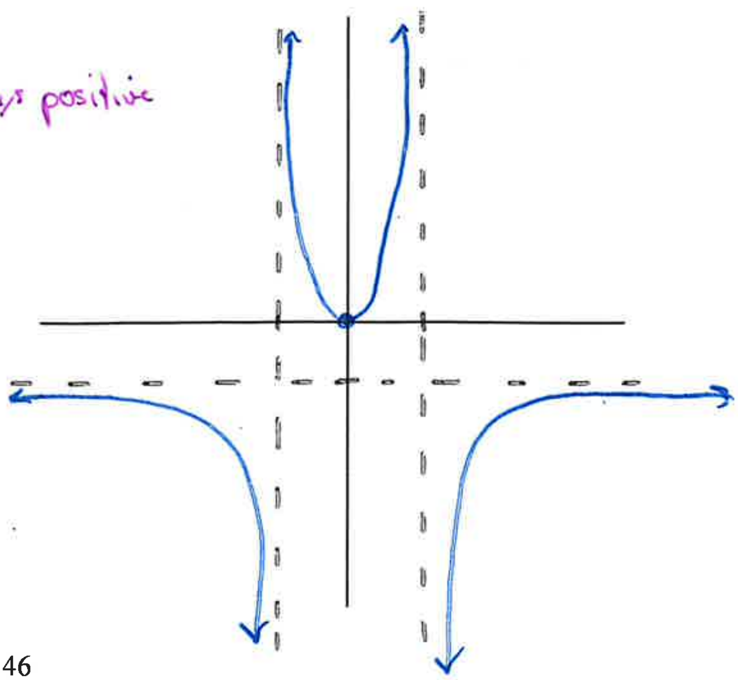
$$= \frac{6x^2 + 2}{(1-x^2)^3} \rightarrow \frac{2(3x^2+1)}{(1-x^2)^3}$$ ← always positive

↑ ppc $x = -1, 1$

Interval	$(3x^2+1)$	$(1-x^2)^3$	
$(-\infty,-1)$	$+$	$-$	CD
$(-1,1)$	$+$	$+$	CU
$(1,\infty)$	$+$	$-$	CD

NO INFLECTION

$x = \pm 1$ not in Domain



Ex. 4 Discuss the curve below and include the headings A-H.

$$y = x\sqrt{2-x}$$

A: Domain $\{x \mid x \in \mathbb{R};\ x \leq 2\}$

B: y-int $(0,0)$

x-int $(0,0)$ $(2,0)$

C: $f(-x) = -x\sqrt{2-(-x)}$

$= -x\sqrt{2+x}$

There is no symmetry

D: Asymptotes

no asymptote

$\lim_{x \to -\infty} = -\infty$

E: $f'(x) = x\left(\frac{1(-1)}{2\sqrt{2-x}}\right) + \sqrt{2-x}$

$= \frac{-x}{2\sqrt{2-x}} + \sqrt{2-x}$

$= \frac{-x + 2(2-x)}{2\sqrt{2-x}}$

$f'(x) = \frac{-x+4-2x}{2\sqrt{2-x}} = \frac{-3x+4}{2\sqrt{2-x}}$

Interval	$-3x+4$	$2\sqrt{2-x}$	$f'(x)$	$f(x)$
$(-\infty, \frac{4}{3})$	+	+	+	inc
$(\frac{4}{3}, 2)$	-	+	-	dec

F: $f(\frac{4}{3}) \doteq 1.09$ local (abs) max

G: $f''(x) = \frac{2\sqrt{2-x}(-3) - (-3x+4)(2)\left(\frac{1(-1)}{2\sqrt{2-x}}\right)}{(2\sqrt{2-x})^2}$

$= \frac{-6\sqrt{2-x} + \frac{2(-3x+4)}{2\sqrt{2-x}}}{4(2-x)}$

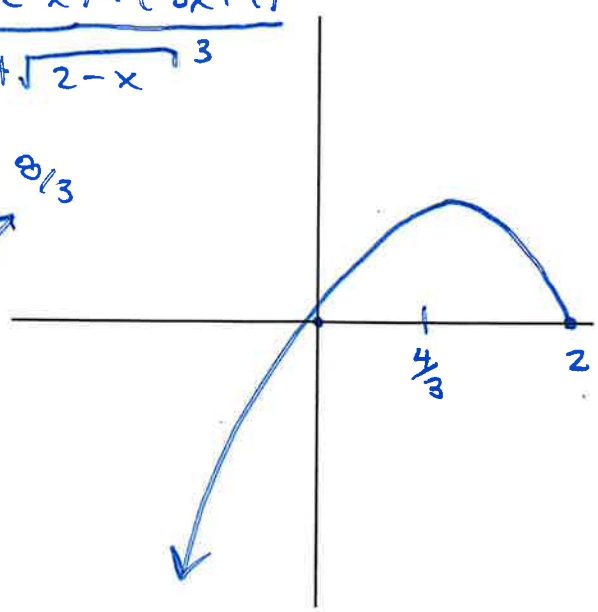
$= \frac{\frac{-12(2-x) + 2(-3x+4)}{2\sqrt{2-x}}}{4\sqrt{2-x}^2} \rightarrow \frac{-6(2-x) + (-3x+4)}{4\sqrt{2-x}^3}$

$\frac{-12+6x-3x+4}{4(2-x)^{3/2}} \rightarrow \frac{3x-8}{4(2-x)^{3/2}} \leftarrow 8/3$

outside domain

CD from $(-\infty, 2)$

no inflection.



Homework Questions

Practice Problems: #1-3