Section 5.5 – Applications of Exponentials and Logarithms

- We look now at solving scenarios where we need exponentials and logarithms
- We have already been introduced to two types of questions that involve exponentials:

Compound Interest

Compound Interest is calculated this way: $\boldsymbol{A = P\left(1 + \frac{r}{n}\right)^{n(t)}}$

- $A: is\ the\ final\ amount\ earned$
- $P: is\ the\ Principal\ (the\ initial\ amount\ of\ money\ borrowed\ or\ saved)$
- $r: is\ the\ \boldsymbol{Yearly}\ Percentage\ Rate, expressed\ as\ a\ decimal\ (25\% = 0.25)$
- n: is the number of times yearly interest is compounded per year
- $t: is\ time, in\ years$

But we have a new scenario: $\boldsymbol{A = Pe^{rt}}$

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Growth/Decay

$A = A_0(x)^{\frac{t}{T}}$	$A = A_0(e)^{kt}$
A – Final Amount A_0 – Initial Amount x – Growth or Decay Value **Examples:** Half-Life: $x = 1/2$ Increase by 10%: $x = 1.1$ Decrease by 10%: $x = 0.9$ t – Total time that item remains T – Time of Growth or Decay	A – Final Amount A_0 – Initial Amount e – Mathematical Constant ≈ 2.71828 k – Proportional Constant t - time

What we have now though, are tools at our disposal to solve for the exponent

Example 1: Vince is considering investing his money. He has $5000 to invest and his broker his giving him two options.

Option 1: 10% interest compounded monthly

Option 2: 10% interest Compounded Continuously

Is there a difference in his choices if he wants to know how long it takes to save $30 000?

Solution 1:

Option 1

$$A = P\left(1 + \frac{r}{n}\right)^{n(t)}$$

$$30000 = 5000\left(1 + \frac{0.10}{12}\right)^{12t}$$

$$6 = 1\left(1 + \frac{0.10}{12}\right)^{12t}$$

$$\log 6 = \log\left(1 + \frac{0.10}{12}\right)^{12t}$$

$$\log 6 = (12t)\log\left(1 + \frac{0.10}{12}\right)$$

$$12t = \frac{\log 6}{\log\left(1 + \frac{0.10}{12}\right)}$$

$$t = \frac{1}{12} \cdot \frac{\log 6}{\log\left(1 + \frac{0.10}{12}\right)}$$

$$t = 17.99\ years$$

Option 2

$$\boldsymbol{A = Pe^{rt}}$$

$$\boldsymbol{30\ 000 = 5000e^{0.1t}}$$

$$\boldsymbol{6 = e^{0.1t}}$$

$$\boldsymbol{\log_e 6 = \log_e e^{0.1t}}$$

$$\boldsymbol{\ln 6 = 0.10t}$$

$$\boldsymbol{t = \frac{\ln 6}{0.10}}$$

$$\boldsymbol{t = 17.92\ years}$$

The options are essentially the same! Take either one Vince.

Example 2: If the half-life of Valeriium-23 is 13 years. How long does it take for 80% of a 5 gram sample to decay?

Solution 2: 80% of $5g$ is $5 \cdot 0.8 = 1g$

We can use either Method for Decay, but remember, we need to solve for k first in the second equation.

$$A = A_0(x)^{\frac{t}{T}}$$

$$1 = 5\left(\frac{1}{2}\right)^{\frac{t}{13}}$$

$$\frac{1}{5} = \left(\frac{1}{2}\right)^{\frac{t}{13}}$$

$$\log_{\frac{1}{2}}\left(\frac{1}{5}\right) = \frac{t}{13}$$

$$\frac{\log 0.2}{\log 0.5} = \frac{t}{13}$$

$$t = 13 \cdot \frac{\log 0.2}{\log 0.5}$$

$$t = 30.2\ years$$

$$A = A_0(e)^{kt}$$

$$2.5 = 5(e)^{k13}$$

$$0.5 = e^{13k}$$

$$\ln 0.5 = 13k$$

$$k = \frac{\ln 0.5}{13}$$

So…

$$1 = 5(e)^{\frac{\ln 0.5}{13}t}$$

$$0.2 = (e)^{\frac{\ln 0.5}{13}t}$$

$$\ln 0.2 = \frac{\ln 0.5}{13}t$$

$$t = \frac{13 \ln 0.2}{\ln 0.5} = 30.2\ years$$

These questions can be challenging, particularly the Growth/Decay questions. Try your best and work through as best you can. Almost always, you will be solving for the exponent, so keep that in mind.

Section 5.5 – Practice Problems

1. On average, a car depreciates 15% in value each year. How long does it take for a new car worth $\$40\ 000$ to depreciate to $\$10\ 000$?

2. If you invest $\$10\ 000$, how long does it take to become a millionaire ($\$1\ 000\ 000$) if you invest in an account at 12% compounded:
   a) quarterly b) continuously

3. What interest rate is needed if money is to triple in 15 years if the interest is compounded:
   a) semi-annually b) continuously

4. It is estimated that 20% of Herlonium decays in 30 hours. What is the half-life of the substance?

5. The pH scale measures the acidity $(0-7)$ or alkalinity $(7-14)$ of a solution with 7 being neutral water. It is a logarithmic scale in base 10. A pH of 9 is 10 times more alkaline than a pH of 8, and a pH of 5 is 10 times more acidic than a pH of 6.

   a) If lemon juice has a pH of 2.1, how many times more acidic is it than black coffee, with a pH of 4.8?
   b) If tomato juice has a pH of 4.2 and is 75 times more acidic than milk. What is the pH of the milk?

6. If Vancouver has a population of $400\ 000$ and is growing at a rate of 2% annually, and Surry has a population of $300\ 000$ and is growing at a rate of 3% annually, in how many years will Surrey catch up to Vancouver in population?

7. Find the time needed for money to triple at 8% compounded:
   a) Daily b) continuously

8. The amount of a chemical in grams that will dissolve in a solution is given by $C = 8e^{0.3t}$ where t is the temperature in Celsius of the solution. Find t when $C = 100\ grams$

9. The population if Toronto is given by $P(t) = 4\ 000\ 000e^{0.012t}, where\ \ t = 0$ corresponds to year 2000. What year will the population reach $6\ 400\ 000$?

10. A biologist studying bacteria determines that a certain culture grows exponentially, such that the colony doubles every 4 days. If the initial colony had 1200 bacteria, how long did it take to have $100\ 000$ bacteria?

11. The half-life of Lukatium is 5570 years. If 500 milligrams are present today, how much is present after 2500 years?

See Website for Detailed Answer Key

Extra Work Space