Section 5.4 – Practice Problems

1. Use the Second Derivative Test to find the local maximum and minimum values of each function, wherever possible.

a) $f(x) = 3x^2 - 4x + 13$

$f'(x) = 6x - 4$

$0 = 6x - 4$

$4/6 = x$

$x = 2/3$ crit #

$f''(x) = 6$

$6 > 0$

local min

$f(2/3) = 35/3$

$\boxed{\left(\frac{2}{3}, \frac{35}{3}\right)}$

b) $f(x) = 2 + 6x - 6x^2$

$f'(x) = -12x + 6$

crit # is $x = \frac{1}{2}$

$f''(x) = -12$

$-12 < 0$ local max

$f(\frac{1}{2}) = \frac{7}{2}$

$\boxed{\left(\frac{1}{2}, \frac{7}{2}\right)}$

c) $g(x) = 2x^3 - 48x - 17$

$g'(x) = 6x^2 - 48$

crit pts

$x^2 = 48/6$

$g''(x) = 12x$

$x = \pm\sqrt{8}$

$g''(\sqrt{8}) = 12(\sqrt{8}) > 0$ local min

$g''(-\sqrt{8}) = 12(-\sqrt{8}) < 0$ local max

$g(\sqrt{8}) \doteq -107.5$ local min

$g(-\sqrt{8}) \doteq 73.5$ local max

d) $g(x) = 1 + 3x^2 - 2x^3$

$g'(x) = 6x - 6x^2$

$= -6x(x-1)$

crit #

0, 1

$g''(x) = 6 - 12x$

$g''(0) = 6$ $\quad 6 > 0$

$g''(1) = -6$ $\quad -6 < 0$

$g(0) = 1$ local min

$g(1) = 2$ local max

e) $h(x) = x^3 - 9x^2 + 24x - 10$

$h'(x) = 3x^2 - 18x + 24$

$= 3(x^2 - 6x + 8) \rightarrow 3(x-4)(x-2)$

crit #: 2, 4

$f''(x) = 6x - 18$

$= 6(x-3)$

$f''(2) = -6 \quad -6 < 0$

$f''(4) = 6 \quad 6 > 0$

$\boxed{h(2) = 10 \text{ local max}}$

$\boxed{h(4) = 6 \text{ local min}}$

f) $h(x) = x^4 - x^3$

$h'(x) = 4x^3 - 3x^2$

$= x^2(4x-3)$

crit #: 0, $3/4$

$h''(x) = 12x^2 - 6x$

$= 6x(2x-1)$

$h''(0) = 0$

$h''(3/4) = 9/4 \quad 9/4 > 0$

$h(3/4) = -27/256$ local min

$h(0) = 0$ not enough info

g) $F(x) = 3x^4 - 16x^3 + 18x^2 + 1$

$F'(x) = 12x^3 - 48x^2 + 36x$

$= 12x(x^2 - 4x + 3) \rightarrow 12x(x-3)(x-1)$

crit #'s: 0, 1, 3

$F''(x) = 36x^2 - 96x + 36$

$F''(0) = 36 \quad 36 > 0$

$F''(1) = -24 \quad -24 < 0$

$F''(3) = 72 \quad 72 > 0$

$F(0) = 1$ local min

$F(1) = 6$ local max

$F(3) = -26$ local min

h) $F(x) = 2 + 5x - x^5$

crit #'s

$x = \pm 1$

$F'(x) = 5 - 5x^4$

$= -5(x^4 - 1)$

$F''(x) = -20x^3$

$F''(1) < 0$

$F''(-1) > 0$

$F(1) = 6$ local max

$F(-1) = -2$ local min

i) $G(x) = (1 - 3x^2 + x^3)^5$

$G'(x) = \underbrace{5(1-3x^2+x^3)^4(-6x+3x^2)}_{\text{use Desmos to find the roots here}}$

OMIT

j) $G(x) = x^2 + 16x^{-1}$

$G'(x) = 2x - 16x^{-2}$

crit #'s: 2

$= \dfrac{2x^3 - 16}{x^2} = \dfrac{2(x^3 - 8)}{x^2}$

$G''(x) = 2 + 32x^{-3}$

$= 2 + \dfrac{32}{x^3}$

$\boxed{G(2) = 12 \text{ local min}}$

$G''(2) = 2 + 4$

$= 6 \quad 6 > 0$

2. Use any method to find the local maximum and minimum values of each function.

a)

$$f(x) = x^4 - 6x^2 + 10$$

$f'(x) = 4x^3 - 12x$
$= 4x(x^2-3)$

crit point
$x = 0$
$x = \pm\sqrt{3}$

$f''(x) = 12x^2 - 12$
$= 12(x^2-1)$
$= 12(x+1)(x-1)$

$f''(0) = -12$ local max

$f''(\sqrt{3}) = 24$
$f''(-\sqrt{3}) = 24$
local min

$\boxed{\begin{array}{l} f(0) = 10 \text{ local max} \\ f(\pm\sqrt{3}) = 1 \text{ local min} \end{array}}$

b)

$$f(x) = x\sqrt{x-1}$$

$f'(x) = x\left(\frac{1}{2\sqrt{x-1}}\right) + \sqrt{x-1} \rightarrow \frac{x}{2\sqrt{x-1}} + \frac{2(x-1)}{2\sqrt{x-1}}$

$= \frac{3x-2}{2\sqrt{x-1}}$

no max/min at $x = 1$
Domain $x > 1$
$x = 2/3$ ← also not in Domain

$\boxed{\text{No max/min}}$

c)

$$g(x) = \frac{x}{x^2+9}$$

$g'(x) = \frac{(x^2+9) - x(2x)}{(x^2+9)^2} = \frac{x^2+9-2x^2}{(x^2+9)^2}$

$= \frac{-(x^2-9)}{(x^2+9)^2}$ crit #'s $x = \pm 3$

Interval	$-(x^2-9)$	$(x^2+9)^2$	$f'(x)$
$(-\infty,-3)$	−	+	−
$(-3,3)$	+	+	+
$(3,\infty)$	−	+	−

← min (at -3), max (at 3)

$\boxed{\begin{array}{l} g(-3) = -\frac{1}{6} \text{ local min} \\ g(3) = \frac{1}{6} \text{ local max} \end{array}}$

d)

$$g(x) = \frac{x}{(2x-3)^2}$$

$g'(x) = \frac{(2x-3)^2(1) - 2(2x-3)(2)x}{((2x-3)^2)^2}$

$= \frac{(2x-3)^2 - 4(2x-3)x}{(2x-3)^4} \rightarrow \frac{2x-3-4x}{(2x-3)^3}$

$= \frac{-2x-3}{(2x-3)^3} = -\frac{(2x+3)}{(2x-3)^3}$

$x = -\frac{3}{2}$
$x = \frac{3}{2}$ ppc

− | + | −
$-\frac{3}{2}$ $\frac{3}{2}$ ← ppc

Does not exist here

$\boxed{f(-\tfrac{3}{2}) = -\frac{1}{24} \text{ local min}}$

e)

$$f(t) = \frac{t^2}{2t+5}$$

$$f'(t) = \frac{(2t+5)2t - t^2(2)}{(2t+5)^2}$$

$$= \frac{4t^2+10t-2t^2}{(2t+5)^2} = \frac{2t^2+10t}{(2t+5)^2}$$

$$= \frac{2t(t+5)}{(2t+5)^2}$$

↑ always positive

Interval	2t	t+5	f'(t)
$(-\infty,-5)$	−	−	+
$(-5,0)$	−	+	−
$(0,\infty)$	+	+	+

$$\boxed{\begin{array}{ll} f(-5) = -5 & \text{local max} \\ f(0) = 0 & \text{local min} \end{array}}$$

f)

$$f(t) = t + 3t^{\frac{2}{3}}$$

$$t = -2^3 = -8$$

$$f'(t) = 1 + 2t^{-\frac{1}{3}}$$

$$= 1 + \frac{2}{t^{1/3}} = \frac{t^{1/3}+2}{t^{1/3}}$$

ppc = 0

Interval	$(t^{1/3}+2)$	$t^{1/3}$	$f'(x)$	$f(x)$
$(-\infty, -8)$	−	−	+	inc
$(-8, 0)$	+	−	−	dec
$(0, 8)$	+	+	+	inc
$(8, \infty)$	+	+	+	inc

max (between first and second rows), min (between second and third rows)

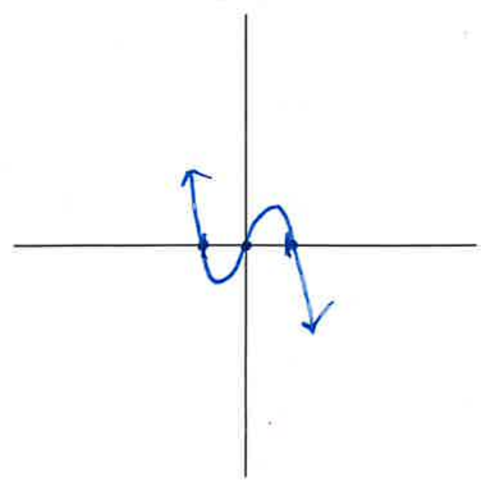
$$\boxed{\begin{array}{ll} f(-8) = 4 & \text{local max} \\ f(0) = 0 & \text{local min} \end{array}}$$

3. Find the maximum and minimum values of each function. Use this information, together with concavity, to sketch the curve.

a) $y = x - x^3$

$$y = -x(x^2-1)$$

$$-x(x+1)(x-1)$$



$$y' = 1 - 3x^2$$

$$-(3x^2-1)$$

$$x = \pm\frac{1}{\sqrt{3}}$$

Interval	$1-3x^2$	
$\left(-\infty, -\frac{1}{\sqrt{3}}\right)$	−	dec
$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	+	inc
$\left(\frac{1}{\sqrt{3}}, \infty\right)$	−	dec

min (between first and second rows), max (between second and third rows)

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{9}$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{2\sqrt{3}}{9}$$

$$f''(x) = -6x$$

$x > 0$ $f''(x) <$ CD

$x < 0$ $f''(x) >$ CU

Inflection pts
$x = \frac{1}{2}$ $y = \frac{15}{16}$
$x = 1$ $y = 1$

b) $y = x^4 - 3x^3 + 3x^2 - x + 1$

Possible factors: $\dfrac{\pm 1}{\pm 4, \pm 1, \pm 2}$

$\pm \frac{1}{4}, \pm 1, \pm \frac{1}{2}$

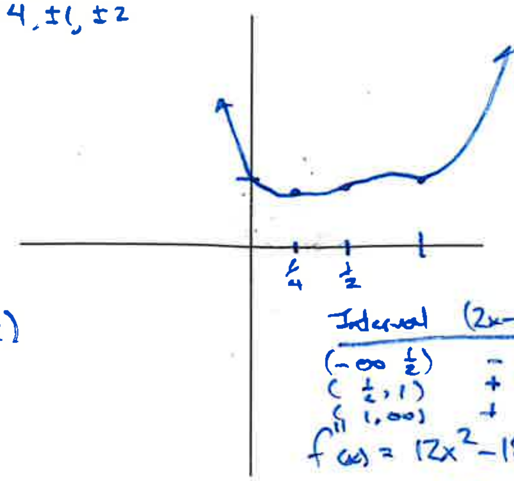
$f'(x) = 4x^3 - 9x^2 + 6x - 1$

$f'(\frac{1}{4}) = 0$

$$\begin{array}{r} x^2 - 2x + 1 \\ 4x - 1 \overline{\big)\, 4x^3 - 9x^2 + 6x - 1} \\ 4x^3 - x^2 \\ \hline -8x^2 + 6x \\ -8x^2 + 2x \\ \hline 4x - 1 \end{array}$$

$(4x-1)(x^2-2x+1)$

$(4x-1)(x-1)^2$



Interval	$(4x-1)$	$(x-1)^2$	$f'(x)$	$f(x)$	
$(-\infty, \frac{1}{4})$	$-$	$+$	$-$	dec	← min
$(\frac{1}{4}, 1)$	$+$	$+$	$+$	inc	
$(1, \infty)$	$+$	$+$	$+$	inc	

$f(\frac{1}{4}) = \frac{229}{256} \doteq 0.9$

Interval	$(2x-1)$	$(x-1)$	$f''(x)$	f
$(-\infty, \frac{1}{2})$	$-$	$-$	$+$	cu
$(\frac{1}{2}, 1)$	$+$	$-$	$-$	cd
$(1, \infty)$	$+$	$+$	$+$	cu

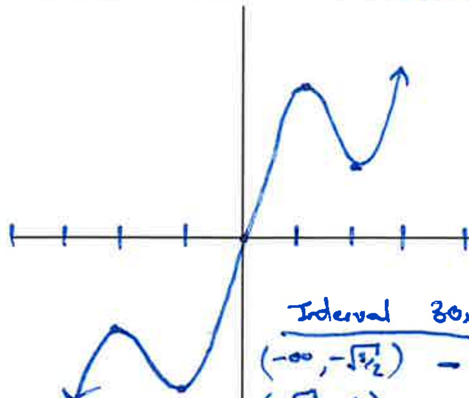
$f''(x) = 12x^2 - 18x + 6$
$= 6(2x^2 - 3x + 1)$
$6(2x-1)(x-1)$

c) $y = 3x^5 - 25x^3 + 60x$

$y' = 15x^4 - 75x^2 + 60$
$= 15(x^4 - 5x^2 + 4)$
$= 15(x^2 - 4)(x^2 - 1)$
$= 15(x+2)(x-2)(x+1)(x-1)$

$y'' = 60x^3 - 150x$
$= 30x(2x^2 - 5)$

crit pts: $x = 0$
$x = \pm\sqrt{\frac{5}{2}}$



crit pts: $\pm 2, \pm 1$

Interval	$(x+2)$	$(x-2)$	$(x+1)$	$(x-1)$	f'	f	
$(-\infty, -2)$	$-$	$-$	$-$	$-$	$+$	inc	
$(-2, -1)$	$+$	$-$	$-$	$-$	$-$	dec	max
$(-1, 1)$	$+$	$-$	0	$-$	$+$	inc	min
$(1, 2)$	$+$	$-$	$+$	$+$	$-$	dec	max
$(2, \infty)$						inc	min

max $f(-2) = -16$
$f(-1) = -38$ min
max $f(1) = 38$
$f(2) = 16$ min

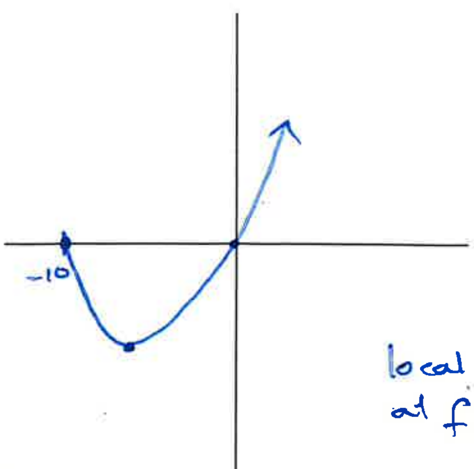
Interval	$30x$	$(2x^2-5)$	$f''x$	$f(x)$
$(-\infty, -\sqrt{5/2})$	$-$	$+$	$-$	cd
$(-\sqrt{5/2}, 0)$	$-$	$-$	$+$	cu
$(0, \sqrt{5/2})$	$+$	$-$	$-$	cd
$(\sqrt{5/2}, \infty)$	$+$	$+$	$+$	cu

d) $y = x\sqrt{10 + x}$

$y' = x\left(\frac{1}{2\sqrt{10+x}}\right) + \sqrt{10+x}$

D: $x > -10$

$= \frac{x}{2\sqrt{10+x}} + \frac{2(10+x)}{2\sqrt{10+x}} \Rightarrow \frac{3x+20}{2\sqrt{10+x}} = \frac{3x+20}{2\sqrt{10+x}}$



crit pt: $-20/3$ if $x < -\frac{20}{3}$ $f'(x) <$ dec
$x > -\frac{20}{3}$ $f'(x) >$ inc

local min
at $f(-\frac{20}{3}) = -\frac{20\sqrt{30}}{9}$

$y'' = \dfrac{2\sqrt{10+x}(3) - \frac{(3x+20)}{\sqrt{10+x}}}{4(10+x)} = \dfrac{2(10+x)(3) - 3x+20}{4(10+x)^{3/2}}$

$= \dfrac{60 + 6x - 3x - 20}{4(10+x)^{3/2}} \rightarrow \dfrac{3x+40}{4(10+x)^{3/2}}$

crit pt $x = -\frac{40}{3}$ ← outside domain

$f''(x) > 0$ cu $(-10, \infty)$