Section 5.4 – Exponential and Logarithmic Relationship

- Using the **exponential and logarithmic relationship** allows use to **solve for variables** when **they are exponents**
- Essentially, **we need to isolate the variable**, like we do in other algebra questions
- But in this case, the **variable may be trapped** in the **exponent** or in the **logarithm**
- **Converting** from one form to the other **allows us to access and isolate it**!
- Recall that in $\log_a x$, $x > 0$, $a > 0$ **and** $a \neq 1$. Anything that **does not follow** those rules**,** becomes an **extraneous solution!**

Example 1: Solve for the unknown

a) $3^x = 81$ b) $2^x = 27$

Solution 1:

$3^x = 81$		$2^x = 27$
$\log 3^x = \log 81$	If $a = b$ then $\log a = \log b$	$\log 2^x = \log 27$
$x \log 3 = \log 81$	Power Rule	$x \log 2 = \log 27$
$x = \dfrac{\log 81}{\log 3}$	Division	$x = \dfrac{\log 27}{\log 2}$
$x = \dfrac{\log 3^4}{\log 3}$	Exponential	$x = \dfrac{\log 3^3}{\log 2}$
$x = \dfrac{4 \log 3}{\log 3}$	Power Rule	$x = \dfrac{3 \log 3}{\log 2} = \dfrac{3(0.4771)}{0.3010}$
$x = 4$	Division	$x = 4.75$

Example 2: Solve for the unknown in: $\log(x-3) + \log x = 1$

Solution 2:

$$\log(x-3) + \log x = 1 \quad \rightarrow \quad \log x(x-3) = 1 \quad \rightarrow \quad x(x-3) = 10^1$$

$$x^2 - 3x - 10 = 0 \quad \rightarrow \quad (x-5)(x+2) = 0 \quad \rightarrow \quad x = 5, -2$$

Reject $\boldsymbol{x = -2}$ since $\boldsymbol{\log x}$ **cannot have a negative object**. Therefore, the solution is: $\boldsymbol{x = 5}$

Example 3: Solve for x: $\log_3(x+12) - \log_3(x+2) = \log_3 x$

Solution 3: $\log_3(x+12) - \log_3(x+2) = \log_3 x$

$\log_3(x+12) = \log_3 x + \log_3(x+2)$	Rearrange the Equation
$\log_3(x+12) = \log_3 x(x+2)$	Product Rule
$(x+12) = x(x+2)$	$x = y$ if $\log_a x = \log_a y$
$(x+12) = x^2 + 2x$	Waterbomb (Distribute)
$x^2 + 2x - x - 12 = 0$	Isolate the Quadratic
$x^2 + x - 12 = 0$	Factor
$(x+4)(x-3) = 0$	Solve
$x = -4, 3$	

Reject $x = -4$ since **$\log x$ cannot have a negative object**. Therefore, the solution is: $\boldsymbol{x = 3}$

Example 4: Solve for x: $2\log_3 x + \log_3(x-1) = 1 + \log_3 2x$

Solution 4: $2\log_3 x + \log_3(x-1) = 1 + \log_3 2x$

$\log_3 x^2 + \log_3(x-1) = 1 + \log_3 2x$	Power Rule
$\log_3 x^2 + \log_3(x-1) - \log_3 2x = 1$	Rearrange the Equation
$\log_3 \frac{x^2(x-1)}{2x} = 1$	Product and Quotient Rule
$\log_3 \frac{x(x-1)}{2} = 1$	Cancel Common Factors
$\frac{x(x-1)}{2} = 3^1$	Convert from Log to Exp
$x(x-1) = 6 \quad \rightarrow \quad x^2 - x - 6 = 0$	Factor
$(x+2)(x-3) = 0$	
$x = -2, 3$	Solve

Reject $x = -2$ since **$\log x$ cannot have a negative object**. Therefore, the solution is: $\boldsymbol{x = 3}$

Example 5: Solve for x: $x^{\log x} = 100x$

Solution 5: $x^{\log x} = 100x$

$\log_x 100x = \log x$	Convert to Log Form
$\dfrac{\log 100x}{\log x} = \log x$	Change of Base
$\log 100x = (\log x)^2$	Multiply the Denominator
$\log 100 + \log x = (\log x)^2$	Product Rule
$(\log x)^2 - \log x - \log 10^2 = 0$	Rearrange and Base 10 Rule
$(\log x)^2 - \log x - 2 = 0$	Write as a Quadratic
$(\log x - 2)(\log x + 1) = 0$	Factor
$\log x = -1 \quad or \log x = 2$	Solve by converting to exponent form
$x = 10^{-1}, 10^2$	

Check both Solutions: $x = 0.1$ and $x = 100$ both works. Solution is: $x = 0.1, 100$

Example 6: Solve for x: $3 \cdot 2^{x-2} = 6^x$

Solution 6: $3 \cdot 2^{x-2} = 6^x$

$\log(3 \cdot 2^{x-2}) = \log 6^x$

$\log 3 + \log 2^{x-2} = \log 6^x$

$\log 3 + (\mathrm{x} - 2)\log 2 = x \log 6$

$\log 3 + \mathrm{x} \log 2 - 2\log 2 = x \log 6$

$x \log 2 - x \log 6 = 2\log 2 - \log 3$

$x \log 2 - x \log 6 = \log 2^2 - \log 3$

$$x = \frac{\log 4 - \log 3}{\log 2 - \log 6}$$

$$x = \frac{\log \frac{4}{3}}{\log \frac{1}{3}} \quad \rightarrow \quad \boldsymbol{x = \frac{\log \frac{4}{3}}{-\log 3}}$$

Example 7: Solve for A in terms of B and C: $5\log A - 2\log B = C$

Solution 7: $5\log A - 2\log B = C \quad \rightarrow \quad \log A^5 - \log B^2 = C$

$\log \frac{A^5}{B^2} = C$

$10^C = \dfrac{A^5}{B^2}$

$10^C \cdot B^2 = A^5$

$\sqrt[5]{10^C \cdot B^2} = \sqrt[5]{A^5}$

$A = \sqrt[5]{10^C \cdot B^2}$

Example 8: If $\log 3 = a \ and \log 8 = b$, determine $\log 18$ in terms of $a \ and \ b$.

Solution 8:

$\log 18 = \log 9 \cdot 2$

$\log 18 = \log 9 + \log 2$

$\log 18 = \log 3^2 + \log 8^{\frac{1}{3}}$

$\log 18 = 2\log 3 + \dfrac{1}{3}\log 8$

$\log 18 = 2a + \dfrac{1}{3}b$

Section 5.4 – Practice Problems

1. Solve for the unknown value. Check for extraneous roots.

a) $\log_5(2x-1)+\log_5(x-2)=1$

b) $\log_2(2-2x)+\log_2(1-x)=5$

c) $\frac{1}{2}-\log_{16}(x-3)=\log_{16}x$

d) $\log_2(3x+1)+\log_2(x-1)=\log_2(10x+14)$

e) $\log_4(3x^2 - 5x - 2) - \log_4(x - 2) = 1$

f) $\log x + \log(29 - x) = 2$

g) $\log_{25}(x - 1) + \log_{25}(x + 3) = \log_7 \sqrt{7}$

h) $2\log(4 - x) - \log 3 = \log(10 - x)$

i) $2\log_2(x+2) - \log_2(3x-2) = 2$

j) $2\log_4 x + \log_4(x-2) - \log_4 2x = 1$

2. Express in the terms stated in each question

a) If $\log x = a$ and $\log y = b$, what is $\log \frac{x^3}{y^2}$ in terms of a and b

b) If $a = \log_2 3$, find $\log_{16} 81$ in terms of a

c) If $\log 3 = a$ and $\log 25 = b$ determine an expression for $\log \frac{9}{5}$ in terms of a and b

d) If $a = \log 2 \ and \ b = \log 3$, what is $\log \frac{25}{9}$ in terms of a and b

e) If $\log A = 2 \ \ and \ \ \log B = 3,$ what is:

i) $\log A/B^2$

ii) $(\log AB)^2$

f) If $\log 2 = a$ and $\log 3 = b$, what is $\log_5 12$ in terms of a and b

g) If $\log AB = 8$ and $\log B = -4$ then what value does $\log A$ equal

h) If $\log 3 = x,\ \log 5 = y, and \log 7 = z,$ then find $\log_2 \sqrt[3]{12.6}$ in terms of x, y, z

i) If $\log_8 3 = a$ and $\log_3 5 = b$, find $\log 5$ in terms of a and b

3. Solve the following

a) Solve for B

$$A = \log 3B - \log C$$

b) Solve for A

$$1 + \log(AB) = \log C$$

c) Solve for A

$$3 \log A + \log B = \log C$$

d) Solve for x

$$\log A = \log B - C \log x$$

4. Solve for x but in terms of logarithmic functions

a) $2^{3x} = 5^{x-1}$

b) $7^{2x-1} = 17^{x}$

c) $3^{x-1} = 9 \cdot 10^{x}$

d) $7^{x-1} = 2 \cdot 5^{1-2x}$

5. Solve for the unknown

a) $\log_2(\log_8 x) = -1$

b) $\log_2(\log_x(\log_3 27)) = -1$

c) $\log_{\frac{1}{2}}(\log_4(\log_2 x)) = 1$

d) $\log x = \frac{2}{3}\log 27 + 2\log 2 - \log 3$

e) $\log x = \log 2 + 3\ \log_{\sqrt{10}} y - \log 2z$

f) $2\log x = -\log a + \log b + 4\log\frac{1}{c}$

6. Determine an equation for the following in terms of logarithmic functions

a) $x = \frac{a^2}{b^3 \cdot c^{\frac{1}{2}}}$

b) $x = \frac{a^{-2}b^3}{c^{-\frac{1}{2}}}$

c) $x = \dfrac{\sqrt[3]{a^2} \cdot b^{-\frac{2}{5}}}{c^{\frac{1}{2}}}$

d) $x = \dfrac{\sqrt{a^5} \cdot b^{-\frac{1}{3}}}{c^3 \cdot d^{-\frac{2}{3}}}$

7. Solve of the unknown value x

a) $\log_2 16^{2x+1} = 8$

b) $\log_{16} x + \log_4 x + \log_2 x = 7$

c) $\log_9 x + 3\log_3 x = 7$

d) $2\log_4 x - 3\log_x 4 = 5$

e) $(\log_4 a)(\log_a 2a)(\log_{2a} x) = \log_a a^3$

f) $\sqrt{\log x} = \log\sqrt{x}$

8. Find the error in the statements written below

a)

$$2 > 1$$

$$\frac{2}{4} > \frac{1}{4}$$

$$\frac{1}{2} > \frac{1}{4}$$

$$\log\frac{1}{2} > \log\frac{1}{4}$$

$$\log\frac{1}{2} > \log\left(\frac{1}{2}\right)^2$$

$$\log\frac{1}{2} > 2\log\frac{1}{2}$$

$$1 > 2$$

b)

$$3 > 2$$

$$3\log\frac{1}{2} > 2\log\frac{1}{2}$$

$$\log\left(\frac{1}{2}\right)^3 > \log\left(\frac{1}{2}\right)^2$$

$$\left(\frac{1}{2}\right)^3 > \left(\frac{1}{2}\right)^2$$

$$\frac{1}{8} > \frac{1}{4}$$

$$1 > 2$$

See Website for Detailed Answer Key

Extra Work Space