Section 5.3 – Properties of Logarithms

- Logarithms have some logic rules to how we can manipulate them
- They are grounded mostly in the exponent laws we learned in grade 9 and 10

Product Rule: $\mathbf{\log_b AB = \log_b A + \log_b B}$

Proof

Let $x = \log_b A$ Let $y = \log_b B$ then….

$b^x = A$ $b^y = B$ so, by exponent laws…

$AB = b^x \cdot b^y \quad \rightarrow \quad AB = b^{x+y}$ and changing to Logarithmic Form…

$\log_b AB = x + y$ then sub back in for x and y

$$\mathbf{\log_b AB = \log_b A + \log_b B}$$

Example 1: Simplify $\log 3 + \log 7$

Solution 1: Since we have the same base of the log, we can use the Product Rule

$$\log 3 + \log 7 \quad \rightarrow \quad \log(3 \cdot 7) = \log 21$$

Quotient Rule: $\mathbf{\log_b \frac{A}{B} = \log_b A - \log_b B}$

Proof

Let $x = \log_b A$ Let $y = \log_b B$ then….

$b^x = A$ $b^y = B$ so, by exponent laws…

$\frac{A}{B} = \frac{b^x}{b^y} \quad \rightarrow \quad \frac{A}{B} = b^{x-y}$ and changing to Logarithmic Form…

$\mathbf{\log_b \frac{A}{B} = x - y}$ then sub back in for x and y

$$\mathbf{\log_b \frac{A}{B} = \log_b A - \log_b B}$$

Example 2: Simplify $\log 36 - \log 4$

Solution 2: Since we have the same base of the log, we can use the Quotient Rule

$$\log 36 - \log 4 \quad \rightarrow \quad \log\left(\frac{36}{4}\right) = \log 9$$

Power Rule: $\boldsymbol{\log_b A^n = n \log_b A}$

Let $x = \log_b A$		then….
$b^x = A$		so, by exponent laws…
$b^{nx} = A^n$	$\rightarrow$ $\log_b A^n = nx$	and changing to Logarithmic Form…
$\boldsymbol{\log_b A^n = nx}$	$\rightarrow$ $\boldsymbol{\log_b A^n = n \log_b A}$	then sub back in for x

Example 3: Simplify $\log_5 125$ using the power rule

Solution 3: Since we have a log, and an object that can be written as a power we can use the Power Rule

$$\log_5 125 \quad \rightarrow \quad \log_5 5^3 \quad \rightarrow \quad 3 \log_5 5 \quad \rightarrow \quad 3$$

$$\log_5 5 = 1$$

If the **base and the object** of the log are **the same** it equals 1

Next, we will see: Change of Base Rule

Remember that the base x in log_x can be any value. Using the change of base rule, you can select what you want the base to be.

If you see:

$\boldsymbol{\log A}$ it is implied that the **base is 10** $\quad$ $\boldsymbol{log_e}$ can be written as the **natural logarithm $\boldsymbol{ln}$**

The Log button on your calculator defaults to Base 10

Change of Base Rule: $\boldsymbol{\log_b a = \frac{\log_x a}{\log_x b}}$

Proof

Let $y = \log_b a$ then….

$b^y = a$ $\quad \log_x b^y = \log_x a$ take the log_x of both sides…

$y \log_x b = \log_x a \quad \rightarrow \quad y = \frac{\log_x a}{\log_x b}$ divide and solve for y, $x > 0$

$\boldsymbol{\log_b a = \frac{\log_x a}{\log_x b}}$ then sub back in for y

<u>Example 4:</u> Find $\log_3 5$ to three decimal places

<u>Solution 4:</u> You need to get to base 10 so you can use the Log button on your calculator

So, when you change the base, pick base 10 (remember, you can pick any base you want)

$$\log_3 5 = \frac{\log 5}{\log 3} = \frac{0.699}{0.477} = 1.465$$

<u>Example 5:</u> Write $\log_3 5$ as a ratio of Natural Logarithms ($\ln$ $(Natural\ Log)$

<u>Solution 5:</u> We are trying to get a log with a base e (a mathematical constant)

You have a Natural Log button on your calculator

$$\log_3 5 = \frac{\log_e 5}{\log_e 3} = \frac{\ln 5}{\ln 3} = \frac{1.606..}{1.099..} = 1.465$$

Good to know rules for Logarithms: In all cases, $b > 0, b \neq 1$		
1. $\log_b 1 = 0$	2. $\log_b b = 1$	3. $\log_b CD = \log_b C + \log_b D$
4. $\log_b \frac{A}{B} = \log_b A - \log_b B$	5. $\log_b A^n = n \log_b A$	6. $\log_b a = \frac{\log_x a}{\log_x b}$

Putting it all Together

- An important aspect of Logarithm rules
- It is solving the puzzle that is ‘what to do’ and ‘what rules can I use’

Example 6: Write each logarithm in terms of $\log 2$ and $\log 5$

a) $\log 40$

b) $\log \frac{125}{4}$

Solution 6: Consider powers and consider factors, always a good place to start

a) $\log 40$

$= \log(8 \cdot 5)$	Factor 40 into strategic factors
$= \log 8 + \log 5$	Product Rule
$= \log 2^3 + \log 5$	Write Object as a Base to a Power
$= \mathbf{3 \log 2 + \log 5}$	Power Rule

b) $\log \frac{125}{4}$

$= \log 125 - \log 4$	Quotient Rule
$= \log 5^3 - \log 2^2$	Write Object as a Base to a Power
$= \mathbf{3 \log 5 - 2 \log 2}$	Power Rule

Example 7: Find the exact value of the following:

a) $\log_5 \sqrt[6]{5}$

b) $\log_3 3^4 - \log_2 2^6$

c) $\log_4 16$

Solution 7:

a) $\log_5 \sqrt[6]{5} \rightarrow \log_5 5^{\frac{1}{6}}$

$\frac{1}{6} \log_5 5 \rightarrow \mathbf{\frac{1}{6}}$

b) $\log_3 3^4 - \log_2 2^6$

$4 \log_3 3 - 6 \log_2 2$

$4 - 6 = \mathbf{-2}$

c) $\log_4 16 \rightarrow \log_4 4^2$

$2 \log_4 4$

$\mathbf{2}$

Simplifying Expressions

- Expanding and condensing Logarithmic expressions is a skill in itself
- An important aspect though, is knowing when you cannot go any further

Example 8: Expand the following Logarithmic Functions

a) $\log_3 5\,x^3y^4$

b) $\log\frac{\sqrt{7x-4}}{5}$

Solution 8:

a) $\log_3 5\,x^3y^4 \quad \rightarrow \quad \log_3 5 + \log_3 x^3 + \log_3 y^4$ Product Rule

$\log_3 5 + 3\log_3 x + 4\log_3 y$ Power Rule

Nothing Else can be Done

b) $\log\frac{\sqrt{7x-4}}{5} \quad \rightarrow \quad \log\sqrt{7x-4} - \log 5$ Quotient Rule

$\log(7x-4)^{\frac{1}{2}} - \log 5$ Root to Exponent

$\frac{1}{2}\log(7x-4) - \log 5$ Power Rule

Nothing Else can be Done

Example 9: Condense each Logarithmic Expression

a) $4\log_2(x+1) + \frac{1}{3}\log_2 x$

b) $\ln 5 - 3\ln(x+1) - 4\ln x$

Solution 9: Watch multiple subtraction, it can help to waterbomb negatives, makes it easier

a) $4\log_2(x+1) + \frac{1}{3}\log_2 x \quad \rightarrow \quad \log_2(x+1)^4 + \log_2 x^{\frac{1}{3}} \quad \rightarrow \quad \log_2(x+1)^4 + \log_2\sqrt[3]{x}$

Exponent Rule

Exponent to Root

$\boldsymbol{\log_2\sqrt[3]{x}(x+1)^4}$

Product Rule

b) $\ln 5 - 3\ln(x+1) - 4\ln x \quad \rightarrow \quad \ln 5 - (3\ln(x+1) + 4\ln x)$ Factor the negative

$\ln 5 - (\ln(x+1)^3 + \ln x^4) \quad \rightarrow \quad \ln 5 - (\ln x^4(x+1)^3)$

Exponent Rule

Product Rule

$$\mathbf{\ln \frac{5}{x^4(x+1)^3}}$$

Other Helpful Rules

More obscure, but helpful rules for Logarithms: In all cases, $b > 0, b \neq 1$		
7. $b^{\log_b a} = a, \quad a > 0$	8. $\log_b a = \frac{1}{\log_a b}$	9. $\log_b a = -\log_{\frac{1}{b}} a$
10. $\log_b \frac{1}{x} = -\log_b x$	11. $\frac{\log_a x}{\log_a y} = \frac{\log_b x}{\log_b y}$	12. $\log_b x = \log_b y$, only when$x = y$

Example 10: Simplify: $\frac{1}{\log_3 12} + \frac{1}{\log_4 12}$

Solution 10:

$\frac{1}{\log_3 12} + \frac{1}{\log_4 12} \quad \rightarrow \quad \log_{12} 3 + \log_{12} 4$ *Rule #8*

$\log_{12} 3 + \log_{12} 4 \quad \rightarrow \quad \log_{12}(3 \cdot 4)$ *Product Rule*

$\log_{12} 12 = 1$ *Rule #2*

- The Change of Base Rule is one of the most helpful rules when simplifying
- It requires a solid understanding of factors and exponential values

Example 11: Simplify $18\log_3 x + 12\log_9 x - 3\log_{27} x$

Solution 11: How do we even approach this? Sometimes play around and see where you end up. But in this case, notice aspects of the base of the log. Each base is a power of 3. Free that base by using Change of Base and start there.

$$18\log_3 x + 12\log_9 x - 3\log_{27} x = \frac{18\log x}{\log 3} + \frac{12\log x}{\log 9} - \frac{3\log x}{\log 27} \quad \boxed{\mathit{Change\ of\ Base}}$$

$$\frac{18\log x}{\log 3} + \frac{12\log x}{\log 9} - \frac{3\log x}{\log 27} = \frac{18\log x}{\log 3} + \frac{12\log x}{\log 3^2} - \frac{3\log x}{\log 3^3} \quad \boxed{\mathit{Law\ of\ Exponents}}$$

$$\frac{18\log x}{\log 3} + \frac{12\log x}{\log 3^2} - \frac{3\log x}{\log 3^3} = \frac{18\log x}{\log 3} + \frac{12\log x}{2\log 3} - \frac{3\log x}{3\log 3} \quad \boxed{\mathit{Power\ Rule}}$$

$$\frac{18\log x}{\log 3} + \frac{12\log x}{2\log 3} - \frac{3\log x}{3\log 3} = \frac{18\log x}{\log 3} + \frac{6\log x}{\log 3} - \frac{\log x}{\log 3} \quad \boxed{\mathit{Divide\ Coefficients}}$$

$$\frac{18\log x}{\log 3} + \frac{6\log x}{\log 3} - \frac{\log x}{\log 3} = \frac{23\log x}{\log 3} \quad \boxed{\mathit{Fraction\ Operations}}$$

$$\mathbf{23\log_3 x} \quad \boxed{\mathit{Reverse\ Change\ of\ Base\ to\ Write\ as\ One\ Cohesive\ Logarithm}}$$

Section 5.3 – Practice Problems

1. Write the following logarithmic expression in terms of $\log 2 \ and \ \log 3$

a) $\log 6$

b) $\log 12$

c) $\log 72$

d) $\log 3200$

e) $\log 0.36$

f) $\log_2 216$

g) $\log 5.4$

h) $\log_6 180$

i) $\log_{18} 2160$

j) $\log_{12} 0.108$

2. Find the exact solution without a calculator

a) $\log_3 81$

b) $\log_2 \frac{1}{32}$

c) $\log_2 \sqrt[4]{8}$

d) $\log_5 \sqrt{125}$

e) $\log_9 27^{2.2}$

f) $\log_4 \frac{1}{32}$

g) $(\log_4 8)(\log_{16} 32)$

h) $\dfrac{\log_{27} 81}{\log_{25} 125}$

i) $\log_4 2 + \log_2 32$

j) $\log_9 16 - 2\log_3 2$

3. Use the properties of logarithms to expand the following (if possible)

a) $\log 100x^2y^3$

b) $\log \frac{x^3}{1000y^2}$

c) $\log(x^2 + y^3)^4$

d) $\log^4(x^2 + y^3)$

e) $\log_5 \frac{25x^2y^3}{z}$

f) $\log\sqrt{x^2(x+2)}$

g) $4\log_2(2x)^{12}$

h) $\log_a \sqrt{\frac{x^2y+1}{a^3}}$

i) $\log\frac{\left(x^3+y\right)^3}{x^3}$

j) $\log\sqrt[3]{\frac{xy^3}{z^6}}$

4. Condense into one single logarithmic quantity

a) $\log_5 x - \log_5 25$

b) $\log_3 x - 2\log_3 27$

c) $\log\sqrt{x} - \log x^{\frac{3}{2}}$

d) $\log(x^2 - 1) - \log(x + 1) - \log x$

e) $\log(3x^2 - 5x - 2) - \log(x^2 - 4) - \log(3x + 1)$

f) $\log_3(2x-3)-\log_3(2x^2-x-3)+\log_3 3(x+1)$

---

g) $2[\log(x^2-1)-\log(x+1)-\log(x-1)]$

---

h) $\frac{3}{2}\log 4x^4-\frac{1}{2}\log y^6$

i) $\frac{1}{4}[\log(x^2-4)-\log(x-2)]-\log x$

j) $\log(x^2-4)-[\log(x-2)+\log(x+2)]$

5. Simplify

a) $\log_b x^{\log_x a}$

b) $x^{\log_x 20-\log_x 4}$

c) $(\log_2 10)(\log 48 - \log 3)$

d) $\dfrac{\log x^3 + \log x^5}{\log x^6 - \log x^3}$

e) $\left(\dfrac{a}{b}\right)^{\log 0.5} \cdot \left(\dfrac{a}{b}\right)^{\log 0.2}$

f) $4^{-2\log_4 3}$

g) $10\log_4 x - 12\log_8 x$

h) $\log\pi + \log\frac{\sqrt{2}}{\pi} + \frac{1}{2}\log\frac{3}{2} - \log\frac{\sqrt{3}}{10}$

i) $\log(1-x^3) - \log(1+x+x^2) - \log(1-x)$

j) $\dfrac{\log_a x}{\log_{ab} x} - \dfrac{\log_a x}{\log_b x}$

---

k) $\dfrac{1}{\log_a x} + \dfrac{1}{\log_b x}$

l) $(\log_5 9)(\log_3 7)(\log_7 5)$

See Website for Detailed Answer Key

Extra Work Space