Section 5.3 – Linear Programming

- Linear inequalities can be used to solve **optimization problems**
- This is where we can find the greatest and least value of given functions
- We call this method **Linear Programming**
- We will deal with two-variable linear programming models, they contain two parts:
    1. **An objective function** tells us the quantity we want to maximize/minimize
    2. The system of constraints consists of linear inequalities whose solutions is called the **feasible solution** with area called the **feasible region**
- The **optimal solution** is at the **vertices** of the feasible regions
- We find the solution by testing the **objective function** at each vertex

Steps to Follow for Solving a Linear Programming Problem

Step 1: Sketch the region R determined by the system of constraints
Step 2: Find the vertices of R
Step 3: Calculate the value of the objective function C at each vertex of R
Step 4: Find the maximum or minimum value(s) of C

Example 1: Find the maximum values of the objective function given by $C = 7x + 3y$, subject to the following constraints: $x - y \geq -4$, $2x + y \leq 10$, $x \geq 0$, $y \geq 0$

$-y \geq -x - 4$ $\quad y \leq -2x + 10$

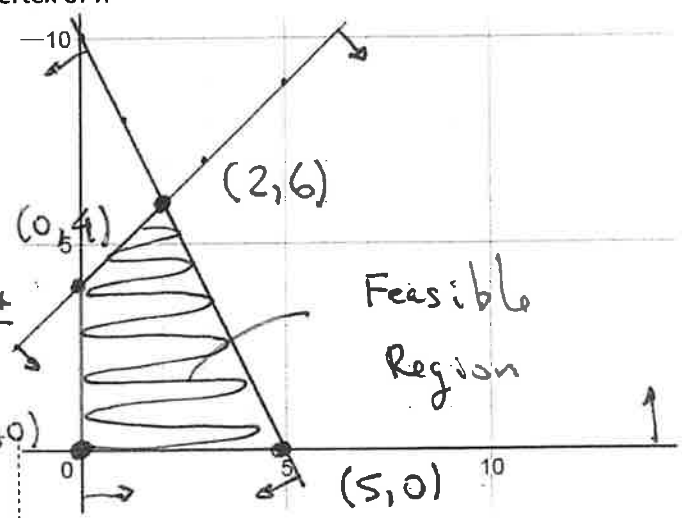
$y \leq x + 4$

Graph the constraints equations using method 2 from Section 6.2

The maximum and minimum values of C must occur at a vertex of R

objective Function

Vertex	Value of $C = 7x + 3y$
$(0, 0)$	$C = 7(0) + 3(0) = 0$ Min
$(0, 4)$	$C = 7(0) + 3(4) = 12$
$(5, 0)$	$C = 7(5) + 3(0) = 35$ Max
$(2, 6)$	$C = 7(2) + 3(6) = 32$



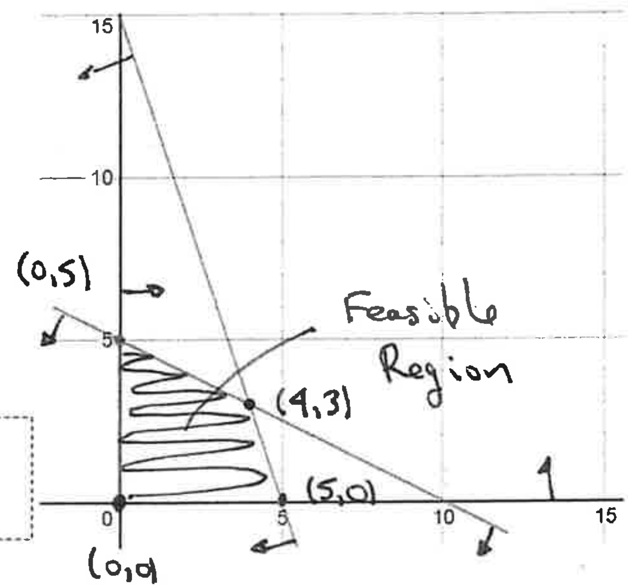
So, the MINIMUM value of C is 0, at $(0, 0)$

And, the MAXIMUM value of C is 35, at $(5, 0)$

Example 2: Maximize and Minimize $C = 3x + 2y$, subject to the following constraints:
$x + 2y \leq 10, \quad 3x + y \leq 15, \quad x \geq 0, \quad y \geq 0$

$2y \leq -x + 10$ $\quad y \leq -3x + 15$

$y \leq -\frac{1}{2}x + 5$



Vertex	Value of $C = 3x + 2y$
$(0,0)$	$C = 3(0) + 2(0) = 0$ Min
$(0,5)$	$C = 3(0) + 2(5) = 10$
$(5,0)$	$C = 3(5) + 2(0) = 15$
$(4,3)$	$C = 3(4) + 2(3) = 18$ Max

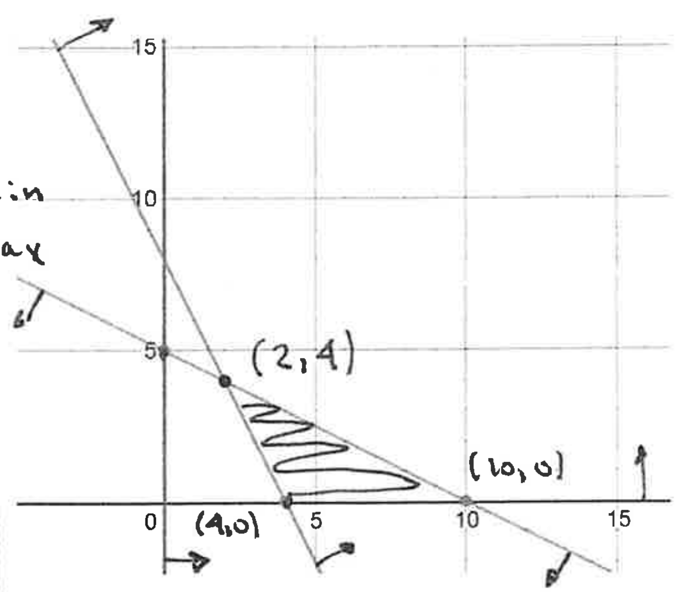
So, the MINIMUM value of C is 0, at $(0,0)$

And, the MAXIMUM value of C is 18, at $(4,3)$

Example 3: Maximize and Minimize $C = 3x + 4y$, subject to the following constraints:
$2x + y \geq 8, \quad x + 2y \leq 10, \quad x \geq 0, \quad y \geq 0$

$y \geq -2x + 8$ $\quad 2y \leq -x + 10$

$y \leq -\frac{1}{2}x + 5$



Vertex	Value of $C = 3x + 4y$
$(2,4)$	$C = 3(2) + 4(4) = 22$
$(4,0)$	$C = 3(4) + 4(0) = 12$ Min
$(10,0)$	$C = 3(10) + 4(0) = 30$ Max

So, the MINIMUM value of C is 12, at $(4,0)$

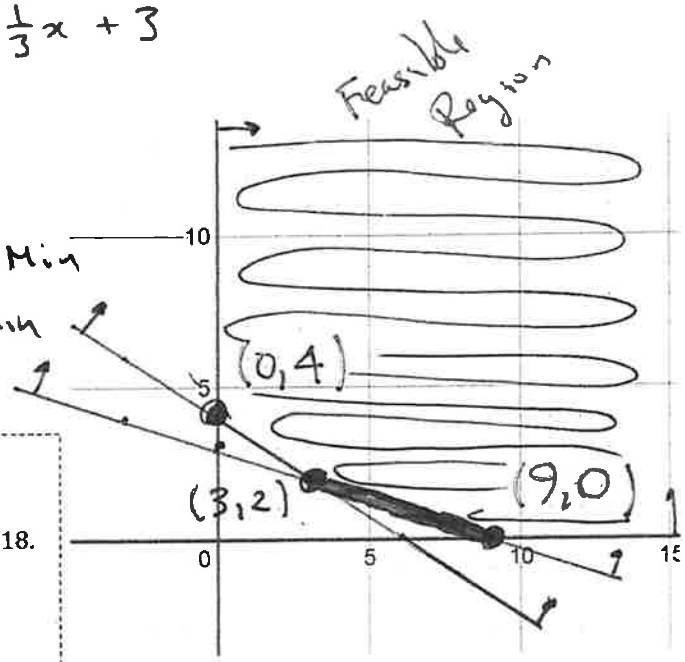
And, the MAXIMUM value of C is 30, at $(10,0)$

Example 4: Maximize and Minimize $C = 2x + 6y$, subject to the following constraints:
$2x + 3y \geq 12, \quad x + 3y \geq 9, \quad x \geq 0, \quad y \geq 0$

$3y \geq -2x + 12 \qquad 3y \geq -x + 9$

$y \geq -\frac{2}{3}x + 4 \qquad y \geq -\frac{1}{3}x + 3$

Vertex	Value of $C = 2x + 6y$
$(0, 4)$	$C = 2(0) + 6(4) = 24$
$(3, 2)$	$C = 2(3) + 6(2) = 18$ Min
$(9, 0)$	$C = 2(9) + 6(0) = 18$ Min



- the MINIMUM value of C *is* **18**, ***at the line segment***: $x + 3y = 9$, *with* $3 \leq x \leq 9$
- Every point on the line between $x = 3$ *and* $x = 9$ *gives* $C = 18$.
- The MAXIMUM value of the 3 vertices is $C =$ **24** but since there **is no upper boundary** to the region there is **no maximum** for the value of this problem

$(3, 2)\ (4, \frac{5}{3})\ (5, \frac{4}{3})$
$(6, 1)\ (7, \frac{2}{3})\ (8, \frac{1}{3})$
$(9, 0)$

Summary of Linear Programming

1. If the linear programming problem has an optimal solution, **either maximum or minimum** of the objective function, then it **occurs at a vertex of the feasible region**
2. If the **feasible region is closed and bounded** (examples 1, 2, and 3), then the objective function has **both a maximum and a minimum**
3. If the objective function has the **same optimum value at two corners**, then the **optimum value is any point on the line segment** connecting the two corner points
4. When the **feasible region is not closed** (example 4), the objective function may have a **maximum only, minimum only, or neither**

Practice Questions #3,7,9.13,15

Section 5.3 – Practice Questions

Determine which of the ordered pairs given produces the maximum values

1. $C = 12x + 10y$
   $(0,0), (7,0), (5,3), (0,8.5)$

2. $C = 50x + 45y$
   $(0,0), (0,21), (15,0), (7.5,12.5)$

3. $C = 16x + 8y$
   $(1,2), (2,1), (0,4), (3,0)$

4. $C = 3x + 5y$
   $(4,3), (1,5), (7,1), (5,2)$

Determine which of the ordered pairs given produces the minimum value

5. $C = 8x + 15y$
   $(0,20), (35,0), (5,15), (12,11)$

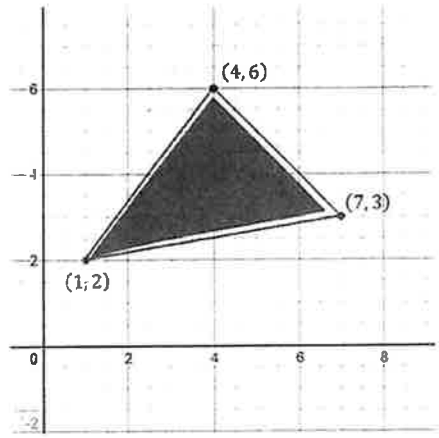
6. $C = 75x + 80y$
   $(0,9), (10,0), (4,5), (5,4),$

7. $C = 3x - 10y$
   $(5,1), (2,0), (10,3), (8,2)$

8. $C = 0.3x - y$
   $(10,1), (20,4), (7,0), (40,11)$

Find the max and min values of the given function and indicated region where they occur

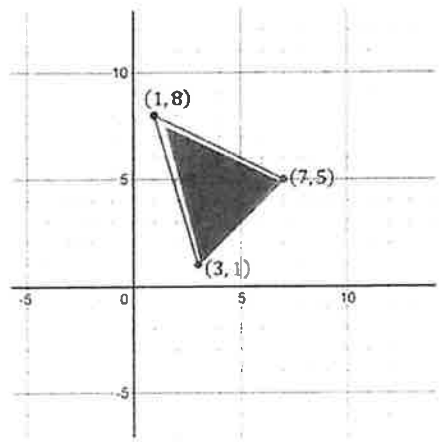
9. $C = 2x - 3y$



Max:

Min:

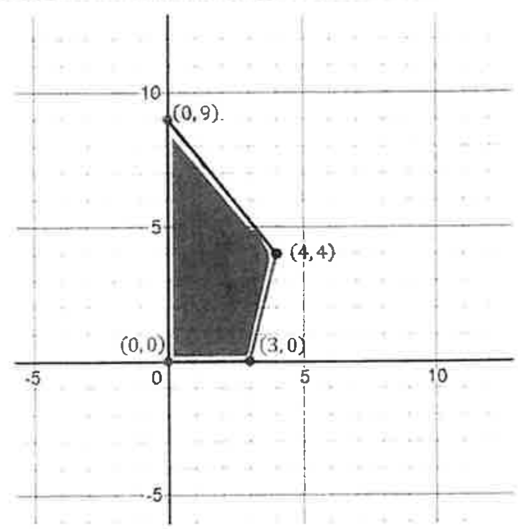
10. $C = x + 3y$



Max:

Min:

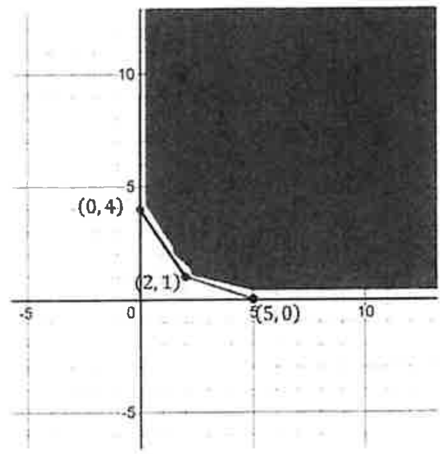
11. $C = 3x - 2y$



Max:

Min:

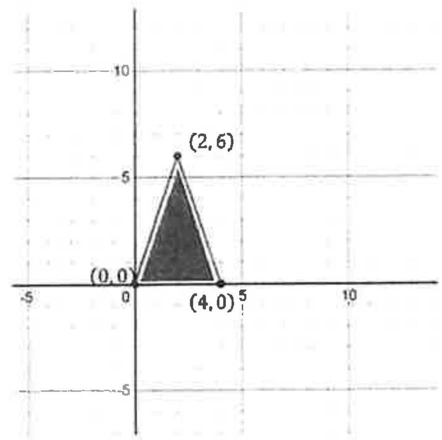
12. $C = 2x + 4y$



Max:

Min:

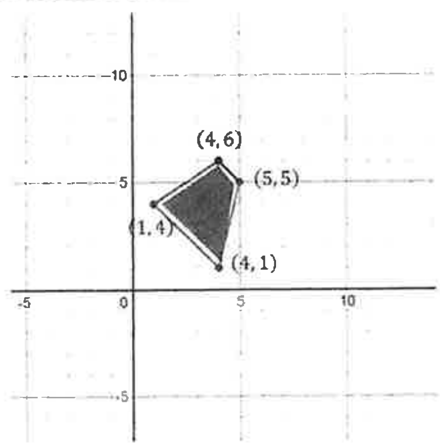
13. $C = 3x + y$



Max:

Min:

14. $C = 3x + 3y$



Max:

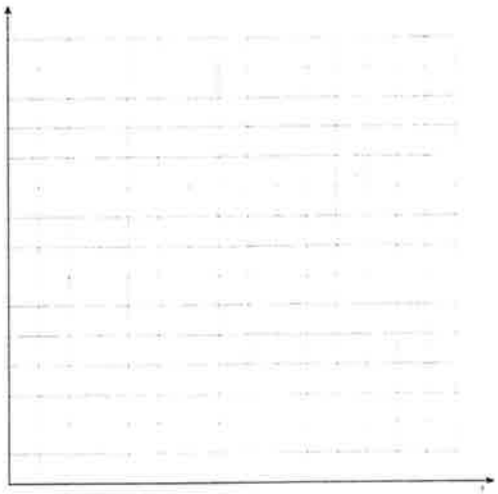
Min:

15. *Maximize* $C = 6x + 4y$

Subject to: $x + 2y \leq 10$

$3x + y \leq 15$

$x \geq 0, y \geq 0$

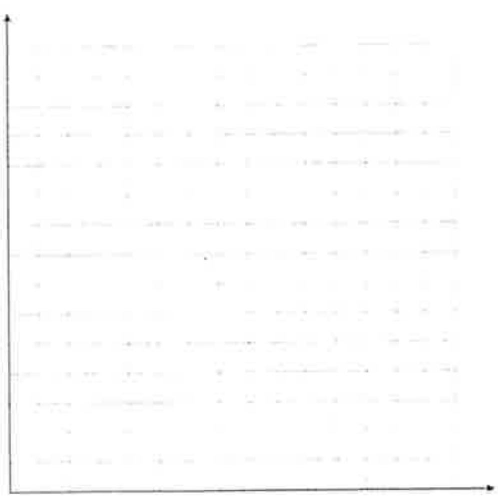


16. *Maximize* $C = 8x + 10y$

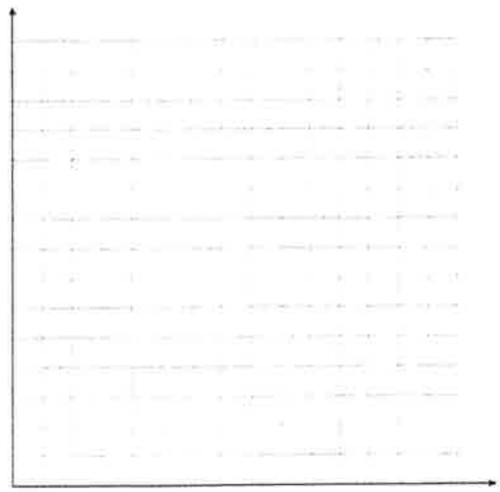
Subject to: $2x + y \leq 12$

$x + 3y \leq 21$

$x \geq 0, y \geq 0$



17. *Minimize* $C = 6x + 8y$
Subject to: $2x + y \geq 8$
$x + 2y \leq 10$
$x \geq 0, y \geq 0$



18. *Minimize* $C = 6x + 3y$
Subject to: $4x + 3y \geq 24$
$4x + y \leq 16$
$x \geq 0, y \geq 0$