Section 5.3 – Practice Problems

1. Find the intervals on which the curve is concave upward or concave downward, and state the points of inflection.

a) $y = 2 + 5x - 12x^2$

$f'(x) = 5 - 24x$

$f''(x) = -24$

↑

always negative

$\boxed{\text{CD} \quad (-\infty, \infty)}$

b) $y = 6x^2 - 12x + 1$

$f'(x) = 12x - 12$

$f''(x) = 12$

↑

always positive

$\boxed{\text{CU} \quad (-\infty, \infty)}$

c) $y = 16 + 4x + x^2 - x^3$

$f'(x) = 4 + 2x - 3x^2$

$f''(x) = 2 - 6x$

$f''(x) = 0 = 2 - 6x$

$6x = 2$

$x = \frac{1}{3}$

+ | −

$\frac{1}{3}$

pt of inflection

$(\frac{1}{3}, \frac{470}{27})$

CU $(-\infty, \frac{1}{3})$

CD $(\frac{1}{3}, \infty)$

d) $y = 2x^3 + 24x^2 - 5x - 21$

$f'(x) = 6x^2 + 48x - 5$

$f''(x) = 12x + 48$

$= 12(x + 4)$

− | +

-4

pt of inflection

$(-4, 255)$

CU $(-4, \infty)$

CD $(-\infty, -4)$

e) $y = x^4 - 2x^3 + x - 2$

$f'(x) = 4x^3 - 6x^2 + 1$

$f''(x) = 12x^2 - 12x$

$= 12x(x-1)$

Interval	$12x$	$(x-1)$	$f''(x)$
$(-\infty, 0)$	$-$	$-$	$+$
$(0, 1)$	$+$	$-$	$-$
$(1, \infty)$	$+$	$+$	$+$

Inflection Pts

$(0, -2)$

$(1, -2)$

CU $(-\infty, 0)$ and $(1, \infty)$

CD $(0, 1)$

f) $y = x^4 - 24x^2 + x - 1$

$f'(x) = 4x^3 - 48x + 1$

$f''(x) = 12x^2 - 48 \rightarrow 12(x^2 - 4)$

$12(x+2)(x-2)$

Inflection Pts

$(-2, -83)$

$(2, -79)$

Int	$(x+2)$	$(x-2)$	$f''(x)$
$(-\infty, -2)$	$-$	$-$	$+$
$(-2, 2)$	$+$	$-$	$-$
$(2, \infty)$	$+$	$+$	$+$

CU $(-\infty, -2)$ and $(2, \infty)$

CD $(-2, 2)$

g) $y = \dfrac{1}{x-1} = (x-1)^{-1}$

$f'(x) = -(x-1)^{-2}$

$f''(x) = 2(x-1)^{-3} \rightarrow \dfrac{2}{(x-1)^3}$

$x = 1$ is a PPC

Interval	$(x-1)^3$	$f''(x)$
$(-\infty, 1)$	$-$	$-$
$(1, \infty)$	[illegible]	[illegible]

CC [illegible] 1 ∞ [illegible]

CD [illegible] $-\infty$ [illegible] 1 [illegible]

Inflection

NONE

$x = 1$ DNE

h) $y = \dfrac{x-2}{5-x}$

$f'(x) = \dfrac{(5-x)(1) - (x-2)(-1)}{(5-x)^2} \rightarrow \dfrac{5 - x + x - 2}{(5-x)^2}$

$= \dfrac{3}{(5-x)^2} \rightarrow 3(5-x)^{-2}$

$f''(x) = -6(5-x)^{-3}(-1)$

$= \dfrac{6}{(5-x)^3}$

PPC $x = 5$

Interval	$(5-x)^3$	$f''(x)$
$(-\infty, 5)$	$+$	$+$
$(5, \infty)$	$-$	$-$

CU $(-\infty, 5)$

CD $(5, \infty)$

NO INFLECTION PT

Inflection
$\left(-\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$
$\left(\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$

CU $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$
CD $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

i)
$$y = \frac{1}{x^2+1} = (x^2+1)^{-1}$$

$$f'(x) = -1(x^2+1)^{-2}(2x) = \frac{-2x}{(x^2+1)^2}$$

$$f''(x) = \frac{(x^2+1)^2(-2) - (-2x)(2(x^2+1))(2x)}{(x^2+1)^4}$$

$$= \frac{-2(x^2+1)^2 + 8x^2(x^2+1)}{(x^2+1)^4} \rightarrow \frac{-2x^2-2+8x^2}{(x^2+1)^3}$$

$$= \frac{6x^2-2}{(x^2+1)^3}$$

$$= \frac{2(3x^2-1)}{(x^2+1)^3}$$

Positive for all x

Interval	$(3x^2-1)$	$f''(x)$
$\left(-\infty, -\frac{1}{\sqrt{3}}\right)$	+	+
$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	−	−
$\left(\frac{1}{\sqrt{3}}, \infty\right)$	+	+

j)
$$y = \frac{1-x^2}{x^3}$$

$$y' = \frac{x^3(-2x) - (1-x^2)(3x^2)}{x^6}$$

$$= \frac{-2x^4 - 3x^2 + 3x^4}{x^6}$$

$$= \frac{x^4 - 3x^2}{x^6}$$

$$y' = \frac{x^2-3}{x^4}$$

$$y'' = \frac{x^4(2x) - (x^2-3)(4x^3)}{x^8}$$

$$= \frac{2x^5 - 4x^5 + 12x^3}{x^8}$$

$$= \frac{-2x^5 + 12x^3}{x^8} = \frac{-2x^2+12}{x^5}$$

$$= \boxed{\frac{-2(x^2-6)}{x^5}}$$

Inflection
$\left(-\sqrt{6}, \frac{5}{\sqrt{6}^3}\right)$
$\left(\sqrt{6}, -\frac{5}{\sqrt{6}^3}\right)$

Interval	x^2-6	x^5	-2	$f''(x)$	
$(-\infty, -\sqrt{6})$	+	−	−	+	CU
$(-\sqrt{6}, 0)$	−	−	−	−	CD
$(0, \sqrt{6})$	−	+	−	+	CU
$(\sqrt{6}, \infty)$	+	+	−	−	CD

k)
$$y = x^{\frac{2}{3}}(5+x)$$

$$y' = x^{2/3}(1) + (5+x)\tfrac{2}{3}x^{-1/3}$$

$$= x^{2/3} + \frac{10}{3}x^{-1/3} + \frac{2}{3}x^{2/3} = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-1/3}$$

$$y'' = \frac{10}{9}x^{-1/3} - \frac{10}{9}x^{-4/3}$$

$$= \frac{10}{9x^{1/3}} - \frac{10}{9x^{4/3}} \rightarrow \frac{10x-10}{9x^{4/3}}$$

$$= \frac{10(x-1)}{9x^{4/3}}$$

Inflection
(1, 6)

Interval	$10(x-1)$	$9x^{4/3}$	$f''(x)$	
$(-\infty, 0)$	−	+	−	CD
$(0, 1)$	−	+	−	CD
$(1, \infty)$	+	+	+	CU

l)
$$y = \frac{x^2}{\sqrt{x+1}}$$

$$y' = \frac{\sqrt{x+1}(2x) - x^2\frac{1}{2}(x+1)^{-\frac{1}{2}}}{x+1}$$

$$y' = \frac{2(x+1)(2x) - x^2}{2(x+1)(\sqrt{x+1})} \rightarrow \frac{4x^2+4x-x^2}{2(x+1)^{3/2}}$$

$$y' = \frac{3x^2+4x}{2(x+1)^{3/2}}$$

cancel out $(x+1)^{\frac{1}{2}}$

$$y'' = \frac{2(x+1)^{3/2}(6x+4) - \cancel{2}\left(\frac{3}{\cancel{2}}\right)(x+1)^{\frac{1}{2}}(3x^2+4x)}{4(x+1)^{6/2}}$$

$$y'' = \frac{(12x+8)(x+1) - 9x^2 - 12x}{4(x+1)^{5/2}}$$

$$= \frac{12x^2+20x+8-9x^2-12x}{4(x+1)^{5/2}} = \frac{\overbrace{3x^2+8x+8}^{\text{always + DESMOS}}}{4(x+1)^{5/2}}$$

Domain of $f(x)$: $x > -1$

only interval $(-1, \infty)$ +

CU on the interval

NO INFLECTION

2. For each of the following functions,

i) ~~a)~~ Find the intervals of increase or decrease
ii) ~~b)~~ Find the local maximum and minimum values
iii) ~~c)~~ Find the intervals of concavity
iv) ~~d)~~ Find the points of inflection
v) ~~e)~~ Sketch the curve

a) $y = 4 - 13x - 6x^2 - x^3$

no roots

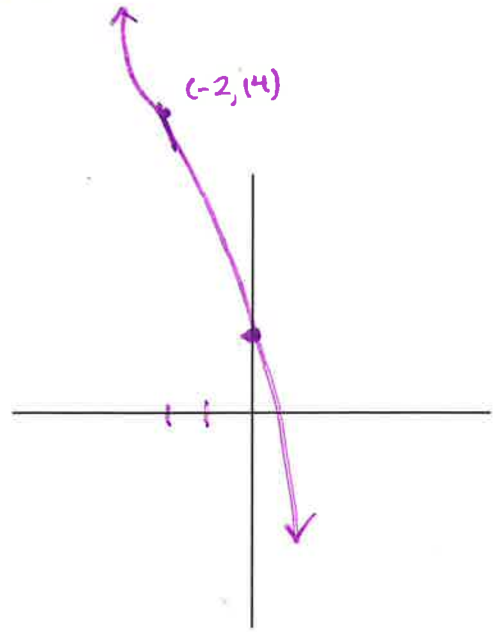
$$\frac{-12 \pm \sqrt{144-4(3)(13)}}{2(3)} = -12 \pm \sqrt{-}$$

i) $y' = -13 - 12x - 3x^2$

$= -3x^2 - 12x - 13 \rightarrow 0 = 3x^2 + 12x + 13$

traditionally opens down

so decreasing always



ii) no max or mins

iii) $y'' = -6x - 12$

$= -6(x+2)$

iv) $\boxed{\text{Inflection at } (-2, 14)}$

Interval	$-6(x+2)$	$f''(x)$
$(-\infty, -2)$	+	CU
$(2, \infty)$	−	CD

y-int: (0, 4)

b) $y = x^4 - 8x^2$

i) $y' = 4x^3 - 16x$

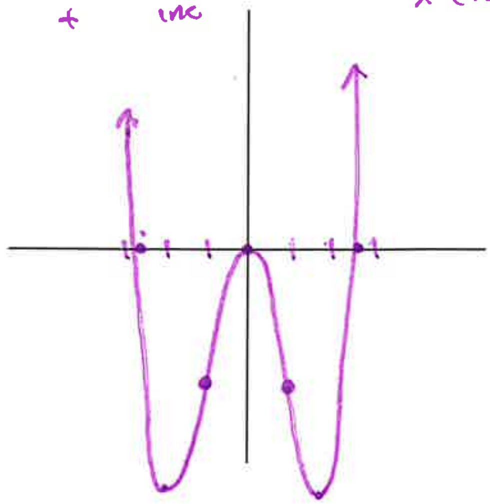
$4x(x^2-4)$

$4x(x+2)(x-2)$

Interval	$4x$	$(x+2)$	$(x-2)$	$f'(x)$	$f(x)$
$(-\infty,-2)$	−	−	−	−	dec
$(-2,0)$	−	+	−	+	inc
$(0,2)$	+	+	−	−	dec
$(2,\infty)$	+	+	+	+	inc

$x^2(x^2-8)$

$x^2(x+\sqrt{8})(x-\sqrt{8})$



ii) crit pts

$f(-2) = -16$ } local min

$f(2) = -16$

$f(0) = 0$ ← local max

iii) $y' = 4x^3 - 16x$

$y'' = 12x^2 - 16$

$= 4(3x^2-4)$

$x = \pm\frac{2}{\sqrt{3}}$

Interval	$(3x^2-4)$	$f''(x)$	$f(x)$
$\left(-\infty, -\frac{2}{\sqrt{3}}\right)$	+	+	CU
$\left(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$	−	−	CD
$\left(\frac{2}{\sqrt{3}}, \infty\right)$	+	+	CU

iv) Inflection Pts

$\left(-\frac{2}{\sqrt{3}}, -\frac{80}{9}\right) \rightarrow (-1.6, -8.9)$

$\left(\frac{2}{\sqrt{3}}, -\frac{80}{9}\right) \quad (1.6, -8.9)$

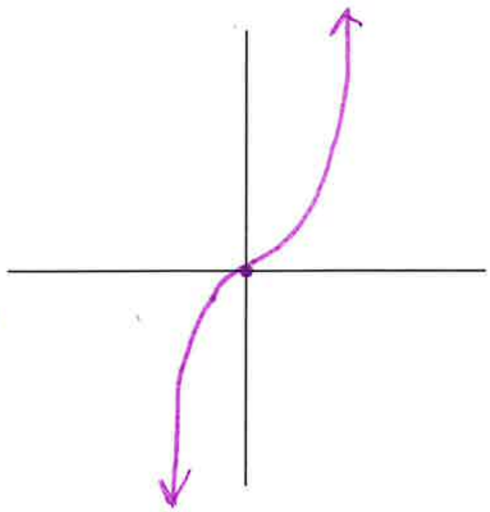
c) $y = x\sqrt{x^2+4}$ roots $x=0$ no domain restriction

i $y' = x\dfrac{1 \cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+4}} + \sqrt{x^2+4}\,(1)$

$= \dfrac{x^2 + x^2+4}{\sqrt{x^2+4}} = \boxed{\dfrac{2x^2+4}{\sqrt{x^2+4}}}$ ← never equal to 0, always positive

i) always increasing

ii) no max or min



iii) $y'' = \dfrac{\sqrt{x^2+4}\,(4x) - (2x^2+4)\dfrac{1 \cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+4}}}{(x^2+4)}$

$\dfrac{\dfrac{4x(x^2+4) - (2x^2+4)x}{\sqrt{x^2+4}}}{(x^2+4)} \rightarrow \dfrac{4x^3+16x-2x^3-4x}{(x^2+4)^{3/2}} = \dfrac{2x^3+12x}{(x^2+4)^{3/2}} \rightarrow \dfrac{2x(x^2+6)}{(x^2+4)^{3/2}}$

no 0 possible

no 0 possible

at pt $x=0$

when $x<0$ $f''(x) < 0$ CD

$x>0$ $f''(x) > 0$ CU

Inflection is at $(0,0)$

d)

$$y = 3x^{\frac{2}{3}} - 2x$$

root $(0,0)$

$y' = 2x^{-\frac{1}{3}} - 2 \rightarrow 2\left(\frac{1}{x^{\frac{1}{3}}} - 1\right)$

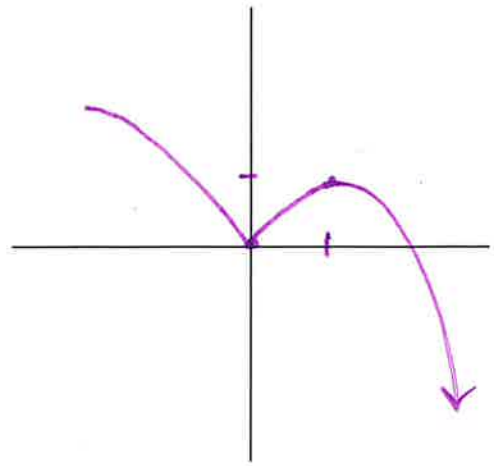
ppc $x = 0$ crit pt. $x = 1$

i)

Interval	$(x^{-\frac{1}{3}} - 1)$	$f'(x)$	$f(x)$
$(-\infty, 0)$	$-$	$-$	dec
$(0, 1)$	$+$	$+$	inc
$(1, \infty)$	$-$	$-$	dec

ii) $f(0) = 0 \leftarrow$ local min

$f(1) = 1 \leftarrow$ local max



iii) $y' = 2x^{-\frac{1}{3}} - 2$

$y'' = -\frac{2}{3}x^{-\frac{4}{3}}$

$= \frac{-2}{3x^{\frac{4}{3}}} \leftarrow$ ppc $x = 0$

Interval	$-\frac{2}{3x^{4/3}}$	
$(-\infty, 0)$	$-$	concave down
$(0, \infty)$	$-$	concave down

no inflection pts

3. For what values of the constants c and d is $(4, -7)$ a point of inflection of the cubic $y = x^3 + cx^2 + x + d$?

$y' = 3x^2 + 2cx + x$

$y'' = 6x + 2c$

$0 = 2(3x+c)$ at $x=4$

$0 = 2(12+c$ $\boxed{c=-12}$

$-7 = 4^3 + (-12)(4)^2 + 4 + d$

$-7 = 64 - 192 + 4 + d$

$\boxed{d = 117}$

4. Show that the function $f(x) = x|x|$ has an inflection point at (0,0), but $f''(0)$ does not exist.

$f(x) = x^2$ if $x > 0$ $\rightarrow f'(x) = 2x$ $\rightarrow f''(0) = 2$ $\leftarrow$ cu

$f(x) = -x^2$ if $x < 0$ $\rightarrow f'(x) = -2x$ $\quad f''(0) = -2$ $\leftarrow$ cd

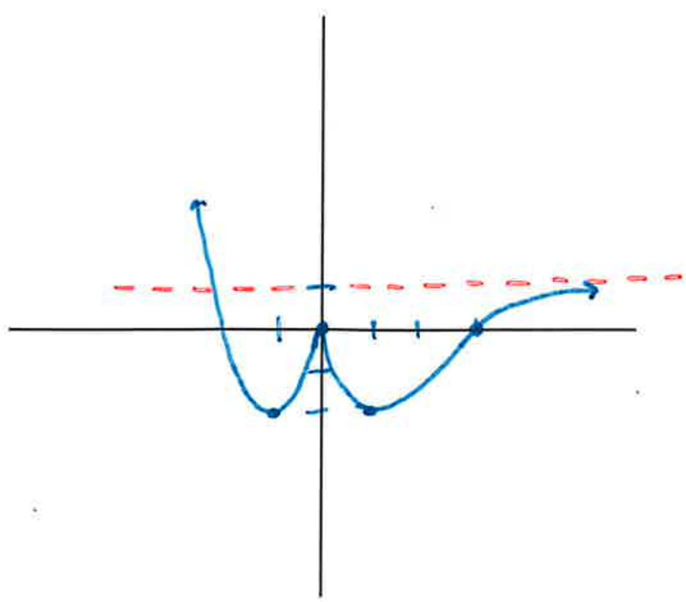
so inflection exists at $f(0) = 0$ but since $\lim_{x \to 0^-} = \lim_{x \to 0^+}$ of $f''(x)$

$f''(x)$ DNE

5. Sketch the graph of a continuous function that satisfies all of the following conditions.

a) $f(0) = f(3) = 0, f(-1) = f(1) = -2$
b) $f'(-1) = f'(1) = 0$
c) $f'(x) < 0$ for $x < -1$ and for $0 < x < 1, f'(x) > 0$ for $-1 < x < 0$ and for $x > 1$
d) $f''(x) > 0$ for $x < 3$ $(x \neq 0), f''(x) < 0$ for $x > 3$
e) $\lim_{x \to \infty} f(x) = 1, \lim_{x \to -\infty} f(x) = \infty$

a) Roots
b) min/max
c) dec/inc intervals
d) concavity and inflection
e) Horizontal asymp



6. Sketch the graph of a continuous function that satisfies all of the following conditions.

   a) $f'(x) > 0$ for $0 < x < 1$, $f'(x) < 0$ for $x > 1$
   b) $f''(x) < 0$ for $0 < x < 2$. $f''(x) > 0$ for $x > 2$
   c) $\lim_{x \to \infty} f(x) = 0$
   d) $f(-x) = -f(x)$ for all x

a) inc/dec

b) concave down $0<x<2$

concave up $x>2$

c) Horizontal asymptote

d) symmetric about the origin