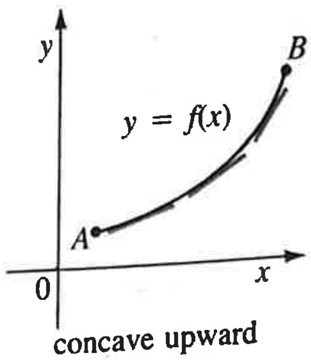
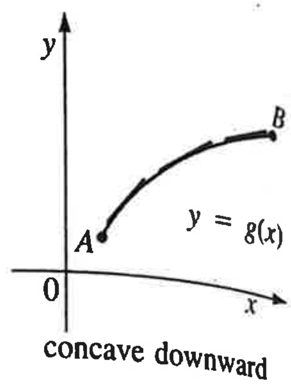
5.3 Concavity and Points of Inflection

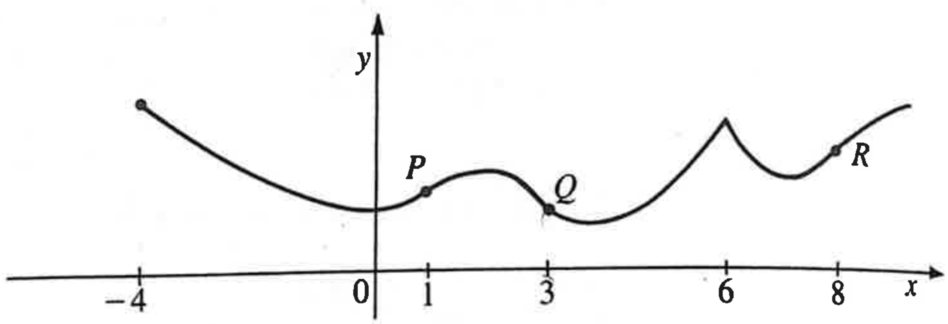
The graphs of the function f and g are shown below. Each graph connects point A to point B, but the bend in different directions, this bending is known as **concavity**. A graph that **lies above its tangent lines** is **concave upward**, and one that **lies below its tangents** is **concave downward**.





In general, the graph of f is considered **concave up** on an interval I if it lies above its tangents over the entire interval. And is considered **concave down** on an interval I if it lies below its tangents over the entire interval.

The graph below shows a function that is concave up (CU) on the intervals $(-4, 1)$, $(3, 6)$, and $(6, 8)$, and is concave down (CD) on the intervals $(1, 3)$ and $(8, \infty)$



At points P, R, Q we see where a function shifts from CU to CD. These are known as points of inflection. Notice that it is at that these points of inflection that the curve crosses its tangent line

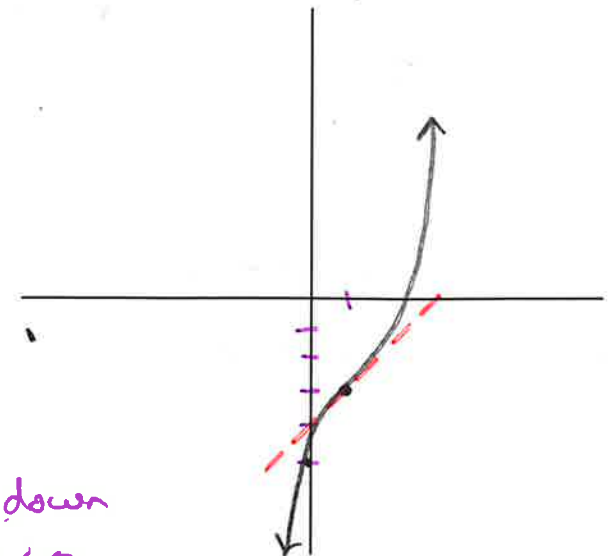
Test for Concavity
If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

It follows that there will be a point of inflection where the second derivative changes sign.

Ex. 1

a) Determine where the curve $y = x^3 - 3x^2 + 4x - 5$ is concave upward and where it is concave downward.
b) Find the points of inflection
c) Use this information to sketch the curve

y-int: $(0,-5)$



a) $y' = 3x^2 - 6x + 4$

$y'' = 6x - 6$

$6(x-1)$

concave up when $y'' > 0$

$6(x-1) > 0$

$x - 1 > 0$

$x > 1$

concave down

$6(x-1) < 0$

$x - 1 < 0$

$x < 1$

CU	CD
$(1,\infty)$	$(-\infty,1)$

b) Point of Inflection

$x = 1$ $\quad y = (1)^3 - 3(1)^2 + 4(1) - 5 \rightarrow -3$ $\quad (1,-3)$

c) $y' = 3x^2 - 6x + 4 \leftarrow$ Quad Eqn $\dfrac{6 \pm \sqrt{36 - 4(3)(4)}}{2(3)}$

$\dfrac{6 \pm \sqrt{-12}}{6} \leftarrow$ does not exist

so y' is always > 0

so always increasing

Ex. 2 Discuss the behaviour of the following function with respect to concavity and points of inflection.

$$y = \frac{x}{x^2+1}$$

$$y' = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2} \rightarrow \frac{-x^2+1}{(x^2+1)^2}$$

$$y'' = \frac{(x^2+1)^2(-2x) - (-x^2+1)(2)(x^2+1)(2x)}{(x^2+1)^4}$$

$$= \frac{(-2x)(x^2+1)^2 - 4x(1-x^2)(x^2+1)}{(x^2+1)^4}$$

$$= \frac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3}$$

$$= \frac{-2x^3-2x-4x+4x^3}{(x^2+1)^3}$$

$$= \frac{2x^3-6x}{(x^2+1)^3} \rightarrow \boxed{\frac{2x(x^2-3)}{(x^2+1)^3}}$$

Inflection Points
$(-\sqrt{3}, -\sqrt{3}/4)$
$(0,0)$
$(\sqrt{3}, \sqrt{3}/4)$

Denominator is always positive

$f''(x) = 0$ when $x = 0, \pm\sqrt{3}$

Interval	$2x$	(x^2-3)	$f''(x)$	$f(x)$
$(-\infty, -\sqrt{3})$	−	+	−	CD
$(-\sqrt{3}, 0)$	−	−	+	CU
$(0, \sqrt{3})$	+	−	−	CD
$(\sqrt{3}, \infty)$	+	+	+	CU

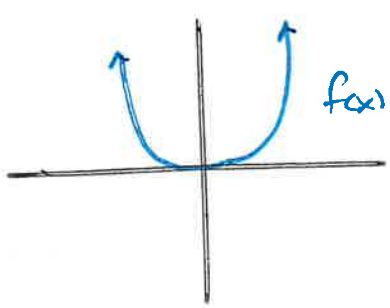
Ex. 3 Show that the function $f(x) = x^4$ satisfies $f''(0) = 0$ but has no inflection points

$f'(x) = 4x^3$

$f''(x) = 12x^2 \leftarrow$ possible inflection points $(0,0)$

but...

$f''(x) \geq 0$ for all x so always Concave Up.

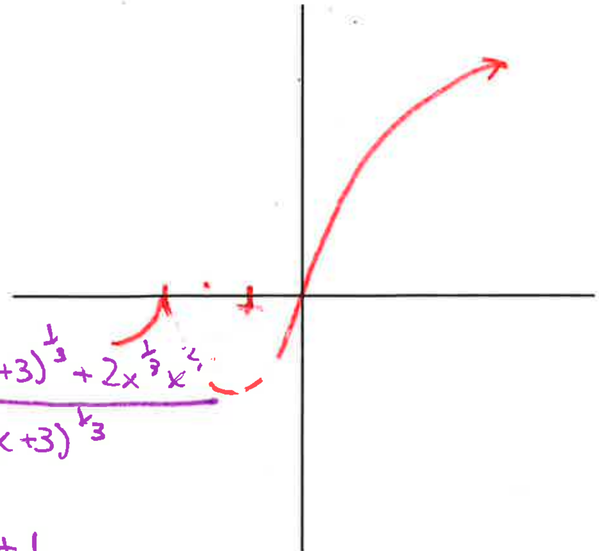


$f(x) = x^4$

Ex. 4 For the function

$$y = x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}$$

a) Find the intervals of increase and decrease
b) Find the local maximum and minimum values
c) Find the intervals of concavity
d) Find the point of inflection
e) Sketch the graph of f



a) $y' = \frac{1}{3}x^{-2/3}(x+3)^{2/3} + \frac{2}{3}(x+3)^{-\frac{1}{3}}(1)(x^{\frac{1}{3}})$

$= \frac{(x+3)^{2/3}}{3x^{2/3}} + \frac{2x^{\frac{1}{3}}}{3(x+3)^{1/3}} \rightarrow \frac{(x+3)^{2/3}(x+3)^{\frac{1}{3}} + 2x^{\frac{1}{3}}x^{\frac{2}{3}}}{3x^{2/3}(x+3)^{1/3}}$

$= \frac{(x+3)+2x}{3x^{2/3}(x+3)^{1/3}} \rightarrow \frac{3(x+1)}{3x^{2/3}(x+3)^{1/3}} = \frac{x+1}{x^{2/3}(x+3)^{1/3}} \longrightarrow$ crit # $x=-1$

ppc $= 0, -3$

b) $f(-3) = 0$ is a local max
$f(-1) \doteq -1.6$ is a local min

	Interval	$(x+1)$	$x^{2/3}$	$(x+3)^{\frac{1}{3}}$	f'
inc →	$-3<x$	−	+	−	+
dec →	$-3<x<-1$	−	+	+	−
inc →	$-1<x<0$	+	+	+	+
inc →	$0<x$	+	+	+	+

c) $f'(x) = \frac{(x+1)}{x^{2/3}(x+3)^{1/3}}$

$f''(x) = \frac{x^{2/3}(x+3)^{\frac{1}{3}}(1) - (x+1)\left[\frac{2}{3}x^{-\frac{1}{3}}(x+3)^{\frac{1}{3}} + \frac{1}{3}(x+3)^{-2/3}x^{2/3}\right]}{x^{4/3}(x+3)^{2/3}}$ ← multiply everything top/bottom by $x^{1/3}(x+3)^{2/3}$

$= \frac{x(x+3) - (x+1)\left[\frac{2}{3}(x+3) + \frac{1}{3}x\right]}{x^{5/3}(x+3)^{4/3}} \rightarrow \frac{x^2+3x-(x+1)(x+2)}{\text{Denom}} = \frac{x^2+3x-x^2-3x-2}{\text{Denom}}$

$= \boxed{\frac{-2}{x^{5/3}(x+3)^{4/3}}}$

↑ Forever positive

$\boxed{f''(x) > 0 \text{ when } x<0 \ \{x \neq -3\};\ f''(x) < 0 \text{ when } x>0}$

Homework Questions

Practice Problems: #1-6

$\boxed{\begin{array}{l}\text{concave up: } (-\infty,-3)\cup(-3,0) \\ \text{concave down: } (0,\infty)\end{array}}$

$\boxed{\begin{array}{l}\text{Inflection pt at} \\ x=0 \quad (0,0) \\ y=0\end{array}}$