Section 5.2 – Logarithms and their Graphs

- Logarithms are the inverse operation of exponentials
- If we flip an exponential over the line $y = x$, we get the general shape of a logarithm
- Logarithmic rules and properties explored in the next few sections allow us to solve information that would otherwise be 'trapped' in an exponential

Since a logarithm (log) is the inverse of an exponential consider this:

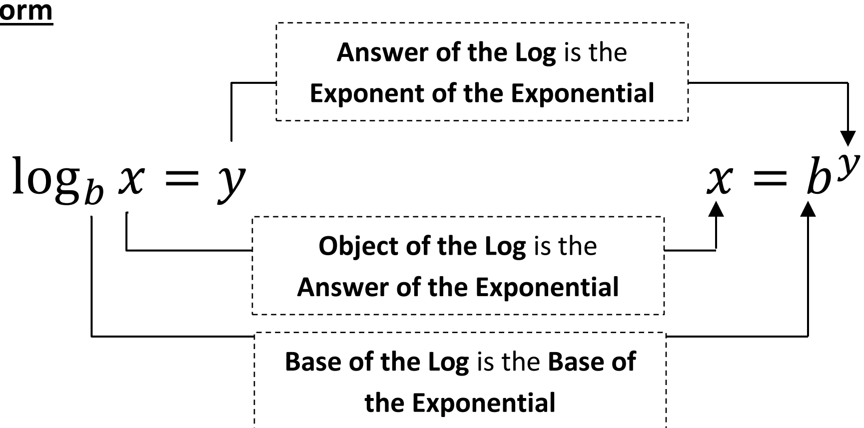
$$y = b^x \quad \text{has an inverse of } x = b^y$$

To free-up the y value we need logarithmic form, the relationship is as follows.

Definition of a Logarithmic Function

$For\ \boldsymbol{b} > \mathbf{0}, b \neq 1$ $\qquad y = \log_b x$ is equivalent to $x = b^y$, $\boldsymbol{x} > \mathbf{0}$

Changing the Form



Customary Notation for the Base of a Logarithm

$\log_n a$ is said to be a Base n Log

Examples:

$\log a$	$\log_2 a$	$\log_e a = \ln a$
No Base means **Base 10**	**Base 2**	**Base e makes a Natural Logarithm (ln)**

Example 1: Change the following from logarithmic to exponential form

a) $\log_4 2 = \frac{1}{2}$

b) $\log_2 \frac{1}{8} = -3$

Solution 1:

c) $\log_4 2 = \frac{1}{2} \rightarrow \quad \mathbf{2 = 4^{\frac{1}{2}}}$

d) $\log_2 \frac{1}{8} = -3 \quad \rightarrow \quad \mathbf{\frac{1}{8} = 2^{-3}}$

Example 2: Change the following from logarithmic to exponential form

a) $3^4 = 81$

b) $3^{-2} = \frac{1}{9}$

Solution 2:

a) $\log_3 81 = 4$

b) $\log_3 \frac{1}{9} = -2$

Example 3: Determine the numerical solution to the following

a) $\log_4 8$

b) $\log_{27} 9$

Solution 3: We will use the equation relationship here, but we will learn another way in the sections ahead.

a) Let $x = \log_4 8$

$$4^x = 8$$
$$2^{2x} = 2^3$$
$$2x = 3$$
$$x = \frac{3}{2}$$

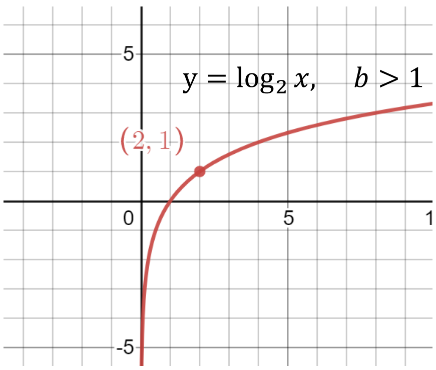
b) Let $x = \log_{27} 9$

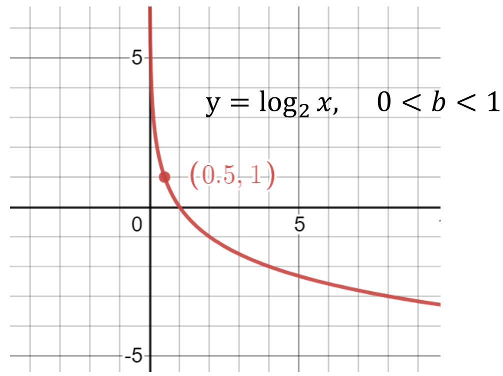
$$27^x = 9$$
$$3^{3x} = 3^2$$
$$3x = 2$$
$$x = \frac{2}{3}$$

Graphs of Logarithmic Functions

Basic Properties of the Graph $\boldsymbol{f(x) = \log_b x, \quad x > 0, \quad b > 0, \quad b \neq 1}$

1. All graphs go through the point $(1, 0)$ and there is no $y - intercept$
2. The $y - axis$ is a vertical asymptote with equation $x = 0$
3. When $b > 1$, $f(x) = \log_b x$ is an increasing function
4. When $0 < b > 1$, $f(x) = \log_b x$ is a decreasing function





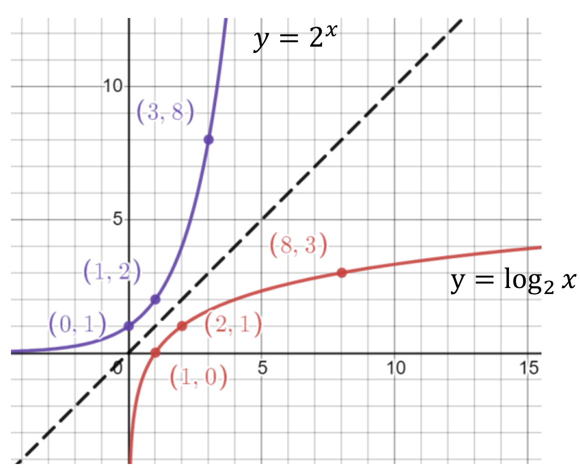
- As mentioned previously, exponentials and logarithms are inverses of one-another
- Here is a proof:

The Statement	**The Proof**
$f(x) = b^x$ and $g(x) = \log_b x$ are inverses	$y = f(x) = b^x \quad \rightarrow \quad x = b^y$ $y = \log_b x \quad \rightarrow \quad f^{-1} = \log_b x$

- And graphically we see that it is a reflection over the line $y = x$.
- And if we look at the Logarithmic Function, the Domain of the Log is the same as the Range of the Exponential

Domain of a Logarithm $\boldsymbol{y = \log_b x}$

$$b > 0, \quad b \neq 1 \qquad \text{and} \qquad x > 0$$

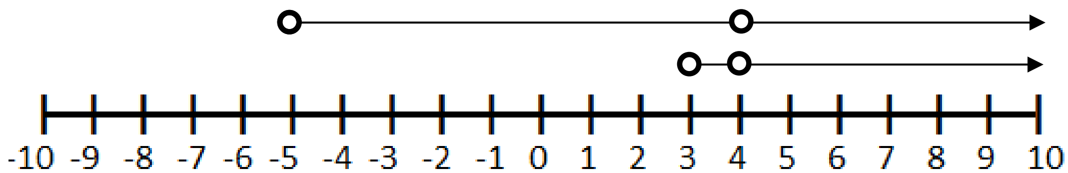


Example 4: Determine the Domain of $y = \log_{x-3}(x+5)$

Solution 4: There are three things to consider here

The Base of the Logarithm first	The Object of the Logarithm second
$x - 3 > 0 \quad \rightarrow \quad x > 3$ $x - 3 \neq 1 \quad \rightarrow \quad x \neq 4$	$x + 5 > 0 \quad \rightarrow \quad x > -5$

Consider this on a number line



We need to consider which stipulations cover all of it: **Domain:** $\boldsymbol{x > 3, \ x \neq 4}$

Example 5: Determine the inverse of $f(x) = 2^{x-1} + 3$

Solution 5: Let $y = f(x)$

$$y = 2^{x-1} + 3 \quad \rightarrow \quad x = 2^{y-1} + 3$$

Swap x and y

$$x - 3 = 2^{y-1} \quad \rightarrow \quad \log_2(x-3) = y - 1$$

Solve for y

$$y = \log_2(x-3) + 1$$

Replace y with $f^{-1}(x)$

$$\boldsymbol{f^{-1}(x) = \log_2(x-3) + 1}$$

Example 6: If the point $(-3, 6)$ is on the graph of $y = b^x$. What point must be on the graph of $y = \log_b x$.

Solution 6: Since the two equations are inverses of one-another, we simply have to swap the coordinates.

$$(-3, 6) \quad \rightarrow \quad (6, -3)$$

Section 5.2 – Practice Problems

1. Write the following in exponential form

a) $\log_4 16 = 2$

b) $\log_3 81 = 4$

c) $\log_6 \frac{1}{36} = -2$

d) $\log \frac{1}{100} = -2$

e) $\log_{32} 8 = \frac{3}{5}$

f) $\log_8 8 = 1$

g) $\log_5 1 = 0$

h) $\log 1000 = 3$

i) $\log_8 4 = \frac{2}{3}$

j) $\log_4 \frac{1}{8} = -\frac{3}{2}$

2. Write the following in Logarithmic Form

a) $2^4 = 16$	b) $8^2 = 64$
c) $16^{\frac{1}{4}} = 2$	d) $3^{-2} = \frac{1}{9}$
e) $3^0 = 1$	f) $10^{-2} = 0.01$
g) $5^1 = 5$	h) $9^{\frac{3}{2}} = 27$
i) $8^{\frac{4}{3}} = 16$	j) $\left(\frac{2}{3}\right)^{-4} = \frac{81}{16}$

3. Evaluate the log without a calculator.

a) $f(x) = \log_2 8$	b) $f(x) = \log_4 16$

c) $f(x) = \log_8 2$

d) $f(x) = \log_{16} 4$

e) $f(x) = \log_5 1$

f) $f(x) = \log_7 7$

g) $f(x) = \log_a a$

h) $f(x) = \log_a a^3$

i) $f(x) = \log_b b^{-4}$

j) $f(x) = \log_5 0$

4. Find the unknow information without a calculator

a) $\log_x 27 = 3$

b) $\log_4 x = -3$

c) $\log 1000 = x$

d) $\log_x 8 = 1$

e) $\log_7 x = -2$

f) $\log_9 27 = x$

g) $\log_x 32 = 2$

h) $\log_4 x = 0$

i) $\log_{32} 8 = x$

j) $\log_x 625 = 4$

k) $\log_4 x = \frac{3}{2}$

l) $\log_4 0.25 = x$

m) $\log_{\sqrt{2}} x = 8$

n) $\log_{\sqrt{3}} x = 4$

o) $\log_x \sqrt{3} = \frac{1}{2}$	p) $\log_{3x} 36 = 2$
q) $\log_{\sqrt{2}} 16 = x$	r) $\log_{\sqrt{3}} 9 = x$
s) $\log_7(x^2 + 24) = 2$	t) $\log(x-2)^2 = -2$

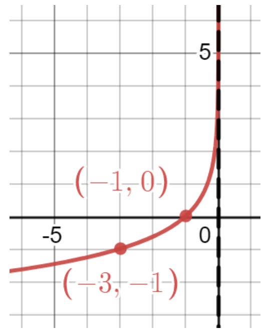
5. What is the Domain of the following functions.

a) $f(x) = \log_3(x-1)$	b) $f(x) = -\log_2 x + 3$
c) $f(x) = \log_{(2-x)} 5$	d) $f(x) = \log_3(-x)$
e) $f(x) = \log_{x+1}(x-2)$	f) $f(x) = \log_{x-2}(x+1)$

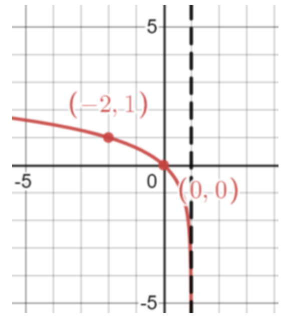
6. Match the Equation to the Graph

a) $f(x) = \log_3(x-1)$	
b) $f(x) = \log_3(1-x)$	
c) $f(x) = \log_3 x + 2$	
d) $f(x) = -\log_3 x$	
e) $f(x) = -\log_3(-x)$	
f) $f(x) = -\log_3(x+2)$	

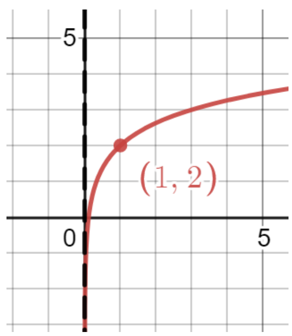
A



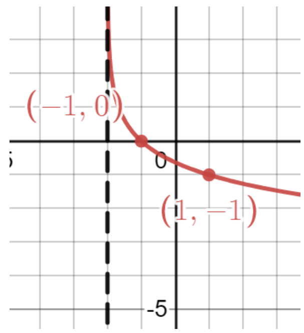
B



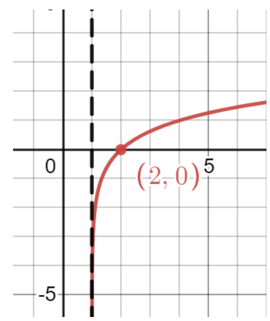
C



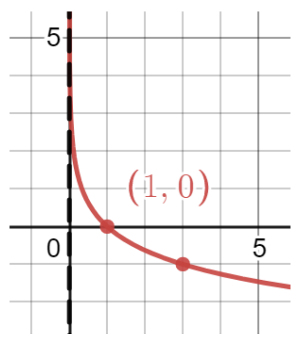
D



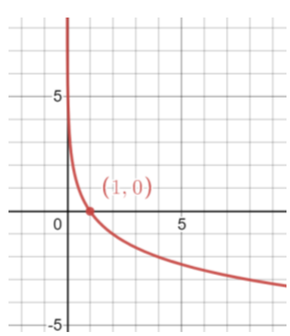
E



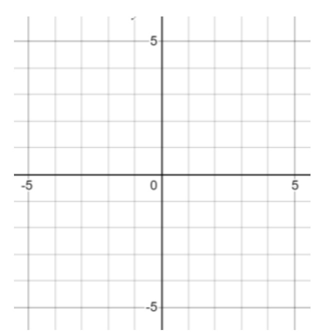
F



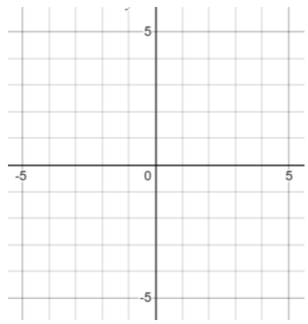
7. If $y = \log_b a$ is shown by the graph below left, what is the shape of:



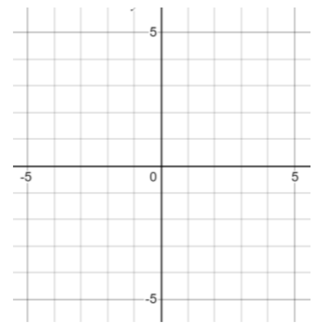
a) $y = -\log_b a$



b) $y = \log_b(-a)$



c) $y = \log_{\frac{1}{b}} a$



8. If point (a, b) is on the graph of $y = 5^x$, what point satisfies $y = \log_5 x$?

9. If point on the graph of $y = \log_2 x$ is $(1,0)$, what point must be on the graph of $y = -\log_2 x$?

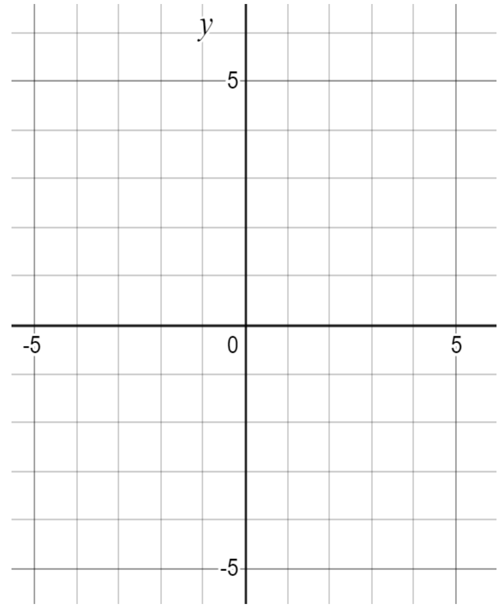
10. If (c, d) is on the graph of $y = \log_b a$ what point must be on the graph of $y = \log_{\frac{1}{b}} a$?

11. Without a calculator, between what two integers do we find:

a) $\log 1253$

b) $\log 0.025$

12. Graph $y = \log(2 - x)$, label asymptotes and axis crossings



13. Determine the inverse of the following functions:

a) $y = 8^{x-2}$

b) $f(x) = 5^{4x-1} + 6$

c) $y + 1 = \log_3(x - 2)$

d) $f(x) = 2 + \log(5x - 3)$

See Website for Detailed Answer Key

Extra Work Space