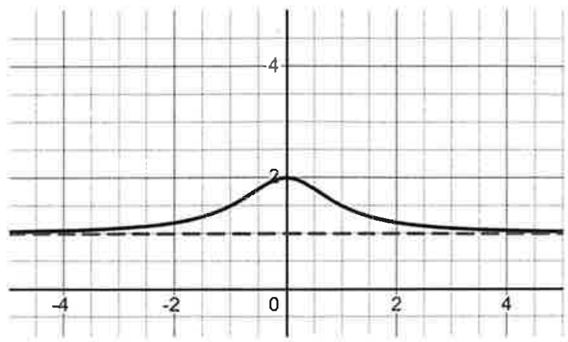
5.2 Horizontal Asymptotes

Consider the behaviour of the following function as x gets large in either direction.

$$f(x) = \frac{x^2+2}{x^2+1}$$

x	$f(x) = \frac{x^2+2}{x^2+1}$
0	2.000 000
± 1	1.500 000
± 2	1.200 000
± 5	1.038 462
± 10	1.009 901
± 100	1.000 100
± 1000	1.000 001



From both the table and thee graph, we can see that as x **gets infinitely large**, our graph approaches the line $y = 1$. We can express this behaviour using our limit notation.

$$\lim_{x \to \infty} \frac{x^2+2}{x^2+1} = 1$$

Likewise, as **decrease our x value infinitely**,

$$\lim_{x \to -\infty} \frac{x^2+2}{x^2+1} = 1$$

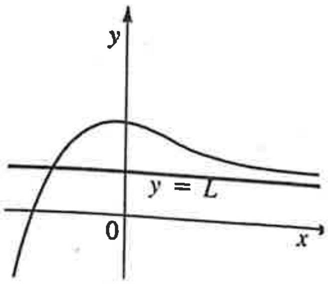
In general, we define a limit at infinity by writing

$$\lim_{x \to \infty} f(x) = L$$

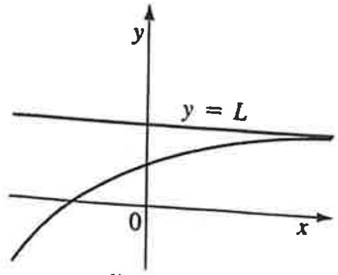
And it is read,

"the limit of $f(x)$, as x approaches infinity, is L"

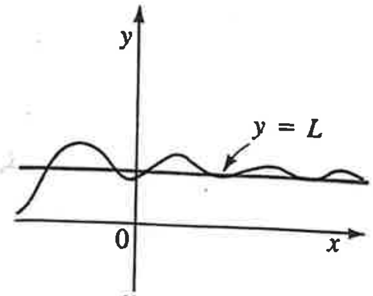
The **line $y = L$** is called a **Horizontal Asymptote** of the curve $y = f(x)$. There are many ways to consider the behaviour of a function, but we **only consider horizontal asymptotic behaviour as x approaches $\pm\infty$.** That means, depending on the scenario, as long as x is not approaching $\pm\infty$ you can in fact cross through a horizontal asymptote.



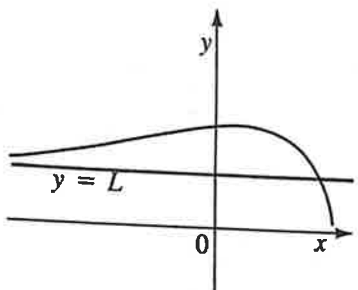
$\lim_{x \to \infty} f(x) = L$



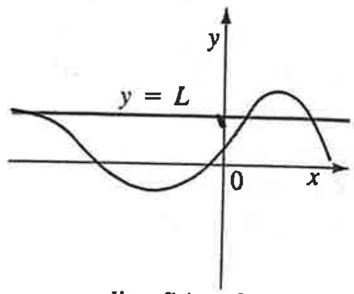
$\lim_{x \to \infty} f(x) = L$



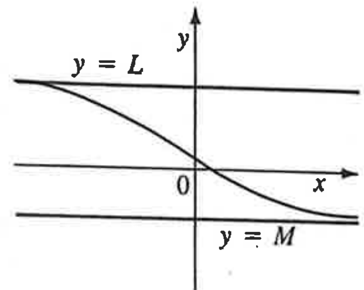
$\lim_{x \to \infty} f(x) = L$



$\lim_{x \to -\infty} f(x) = L$



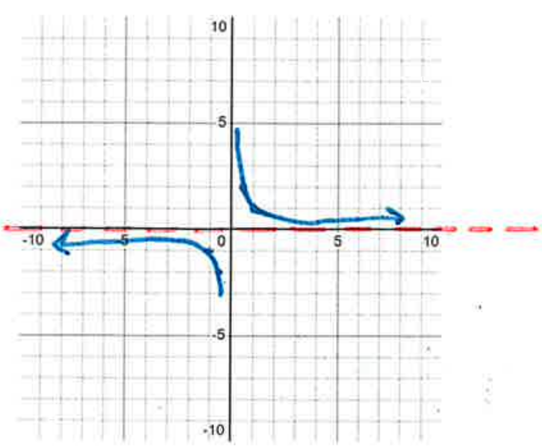
$\lim_{x \to -\infty} f(x) = L$



$\lim_{x \to -\infty} f(x) = L, \lim_{x \to \infty} f(x) = M$

Ex.1 Find

$\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$



If r is a positive rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If r is a positive rational number and x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Ex. 2 Evaluate

$$\lim_{x \to \infty} \frac{4x^2 - x + 2}{6x^2 + 5x + 1}$$

To evaluate a limit of a rational function at infinity, **divide the terms in the numerator and the denominator by the highest power of x that occurs.** We can assume that $x \neq 0$ as we are only interested in large values of x.

highest power is x^2

$$\frac{\frac{4x^2}{x^2} - \frac{x}{x^2} + \frac{2}{x^2}}{\frac{6x^2}{x^2} + \frac{5x}{x^2} + \frac{1}{x^2}} \rightarrow \frac{4 - \frac{1}{x} + \frac{2}{x^2}}{6 + \frac{5}{x} + \frac{1}{x^2}} \quad \lim_{x \to \infty} \rightarrow \frac{4 - \frac{1}{\infty} + \frac{2}{\infty}}{6 + \frac{5}{\infty} + \frac{1}{\infty}}$$

$$\frac{4 - 0 + 0}{6 + 0 + 0} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

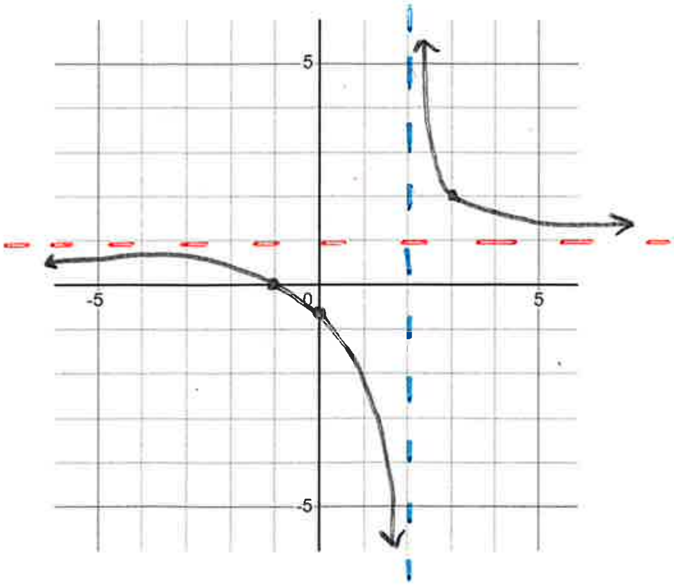
Ex. 3 Find the horizontal and vertical asymptotes of the following function, discuss behaviour at the vertical asymptotes and sketch the graph.

$$y = \frac{x+1}{x-2}$$

VA $\Rightarrow$ $x = 2$

HA $\Rightarrow$ $y = 1$

$$\lim_{x \to \infty} \frac{\frac{x}{x} + \frac{1}{x}}{\frac{x}{x} - \frac{2}{x}} \rightarrow \frac{1+0}{1-0} = \boxed{1}$$



y-int: $-\frac{1}{2}$

x-int: -1

if $x = 3$

$y = 4$

Ex. 4 Find the horizontal and vertical asymptotes of the following function and sketch the graph.

$$y = \frac{x}{x^2 - x - 6}$$

$$\frac{x}{(x-3)(x+2)}$$

y-int: 0

x-int: 0

VA: $x = 3$

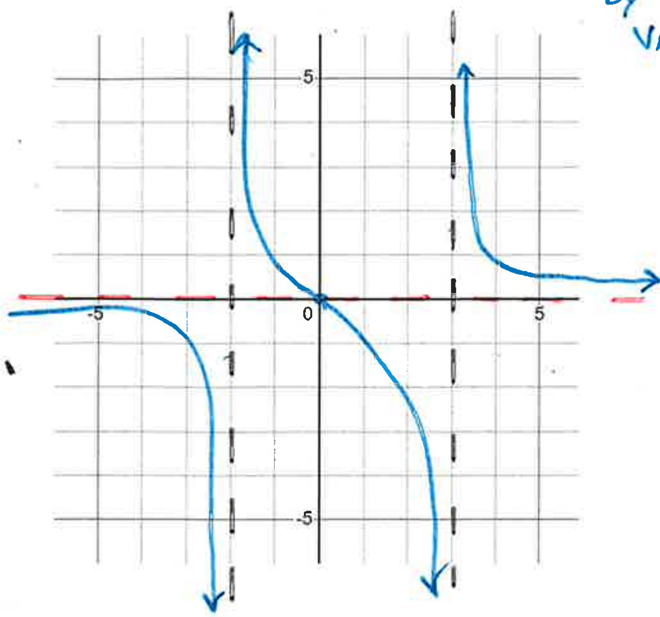
$x = -2$

HA:

$$\lim_{x \to \infty} \frac{\frac{x}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{6}{x^2}} \rightarrow \frac{\frac{1}{\infty}}{1 - \frac{1}{\infty} - \frac{6}{\infty}}$$

$$\frac{0}{1} = 0$$

* Notice we can cross the HA as x is not going to $\pm\infty$, trapped by the VA's

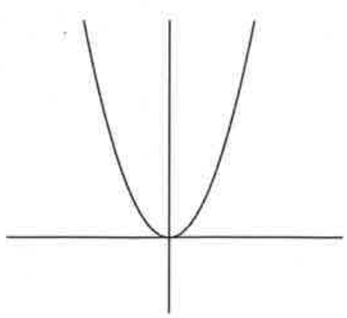


Infinite Limits at Infinity

If $f(x)$ values become infinity large as x does, we use the following notation.

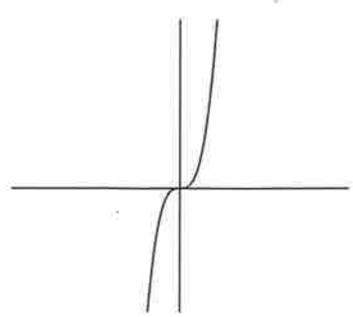
$$\lim_{x \to \infty} f(x) = \infty$$

Similarly, depending on graph behaviour we can see the following:



$$\lim_{x \to \infty} f(x) = \infty$$

$$\lim_{x \to -\infty} f(x) = \infty$$



$$\lim_{x \to \infty} f(x) = \infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

Ex. 5 Find

$$\lim_{x \to \infty} (x^4 - x)$$

We cannot look at this like $\lim_{x\to\infty} x^4 - \lim_{x\to\infty} x$ because ∞ is not a number and $(\infty - \infty)$ cannot be defined.

but..

$\lim_{x\to\infty} x(x^3-1)$ gives $\infty(\infty)$ the product of two infinitely large numbers is infinitely large. $\boxed{\infty}$

Ex. 6 Sketch the graph of $y = (x-3)^2(x+2)(1-x)$ by finding the intercepts and its limits as $x \to \pm\infty$

intercept at 3 but bounces off

intercept at -2

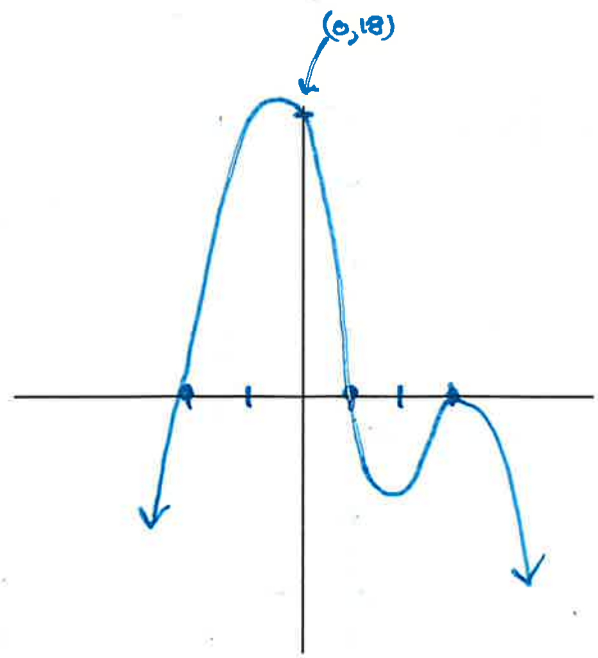
intercepts at 1

4^{th} degree starts down ends down.

y-int: 18

$\lim_{x\to -\infty} = -\infty$

$\lim_{x\to \infty} = -\infty$



Homework Questions

Practice Problems: # 2acfh, 3ad, 4acf, 11