Section 5.1 – Exponents

- In order to study Logarithms (the inverse of exponential growth) we need to first review or exponent laws
- Recall that in grade 9 and 10 we learned…

1. $x^0 = 1$	2. $x^a \cdot x^b = x^{a+b}$	3. $\dfrac{x^a}{x^b} = x^{a-b}$
4. $(x^a)^b = x^{a \cdot b}$	5. $x^{-a} = \dfrac{1}{x^a}$	6. $\left(\dfrac{x}{y}\right)^{-a} = \left(\dfrac{y}{x}\right)^a = \dfrac{y^a}{x^a}$
7. $(xy)^a = x^a y^a$	8. $x^a = x^b$ if and only if $a = b$	

Example 1: Simplify $\dfrac{4^{3x+7}}{8^{2x+5}}$

Solution 1: Notice that **both 4 and 8** can be written as **powers of base 2**

$$\frac{4^{3x+7}}{8^{2x+5}} = \frac{(2^2)^{3x+7}}{(2^3)^{2x+5}} = \frac{2^{2(3x+7)}}{2^{3(2x+5)}} = \frac{2^{6x+14}}{2^{6x+15}} = 2^{6x+14-(6x+15)} = 2^{6x+14-6x-15} = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$$

Example 2: Solve for x when $9^{2x-5} = 27^{2-x}$

Solution 2: Notice that **both 9 and 27** can be written as **powers of base 3**

$$9^{2x-5} = 27^{2-x} \quad \rightarrow \quad 3^{2(2x-5)} = 3^{3(2-x)} \quad \rightarrow \quad 3^{4x-10} = 3^{6-3x}$$

Since the base is the same, we can only compare the exponents (Rule 8)

$$4x - 10 = 6 - 3x \quad \rightarrow \quad 7x = 16 \quad \rightarrow \quad x = \frac{16}{7}$$

Graphing Exponentials

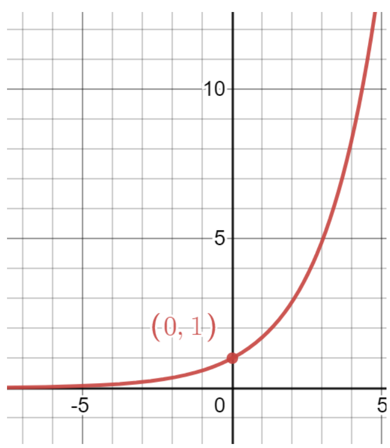
- An exponential function is of the form:

$f(x) = a^x$, where a is a positive number $a > 0$, $a \neq 1$ and x is any real number

Things look a little different if we compare a as a whole number versus a proper fraction

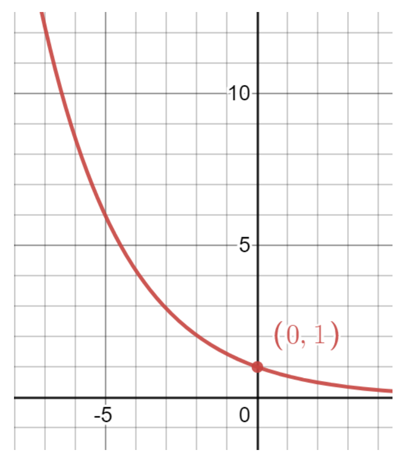
$a^x, a > 1$

Here the **base** gets **exponentially large**



$a^x, 0 < a < 1$ (A proper fraction)

Here the **base is the denominator**, makes the number really small



Both graphs, an **any exponential** passes through the $\boldsymbol{y-int}$ **at** $(\mathbf{0}, \mathbf{1})$ because we $\boldsymbol{x}$ $(\boldsymbol{the\ exponents})$ **is zero the base is 1**

- Exponential graphs have Transformations too (yes, they're back)
- The behaviours are incredibly similar as the general behaviour we saw in Section 2
- Here are some basic characteristics of Exponentials and some transformations

Horizontal Asymptote

The horizontal **line** $\boldsymbol{y = k}$ is the asymptote when $\boldsymbol{f(x)}$ in the graph of $f(x) = a^x$ **approaches** $\boldsymbol{k}$ **as** $\boldsymbol{x \to \pm\infty}$

Basically, a horizontal asymptote appears as the $x\ value$ of the exponential gets very large or very small

Basic Characteristics: $\boldsymbol{f(x) = a^x, a > 0, a \neq 1}$

- ✓ All graphs have $\boldsymbol{y - int}$ $(\mathbf{0}, \mathbf{1}), \boldsymbol{no\ x - int}$
- ✓ $\boldsymbol{x - axis}$ is the $\boldsymbol{horizontal\ asymptote}$
- ✓ When $\boldsymbol{a > 1}$ the function is **increasing**
- ✓ When $\boldsymbol{0 < a < 1}$ the function is **decreasing**

Example 3: Sketch the following graphs:

a) $f(x) = \left(\frac{1}{4}\right)^x - 3$ b) $h(x) = -5^{x-3} + 3$

Solution 3:

a) Let's think transformations, what is happening here?

$f(x) = \left(\frac{1}{4}\right)^x - 3 \quad \rightarrow \quad 4^{-x} - 3$

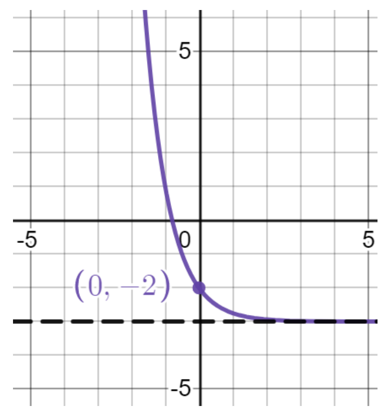
Vertical shift, down 3

See the asymptote shift!

Decreasing Function

Reflection of the $x - values$ (in the $y - axis$

Increasing all of a sudden?? But…



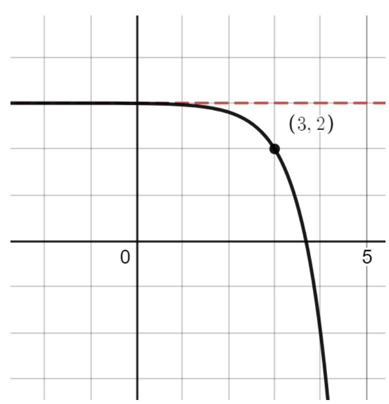
b) $h(x) = -5^{x-3} + 3$

3. Vertical shift, up 3

See the asymptote shift!

1. Reflection of the $y - values$ (in the $x - axis$)

2. Horizontal Shift, see the point $(0,1)$ move three units right



1. $(0, 1) \quad \rightarrow \quad (0, -1)$
2. $(0, -1) \quad \rightarrow \quad (3, -1)$
3. $(3, -1) \quad \rightarrow \quad (3, 2)$

Applications of Exponential Functions

- The Covid Pandemic is a prime example of exponential growth
- Other areas of life we see exponentials and logarithms include radioactive decay, bacteria growth, and compound interest
- Here is an example of how compound interest involves exponentials

Compound Interest is calculated this way: $\boldsymbol{A = P\left(1 + \frac{r}{n}\right)^{n(t)}}$

- $A: \mathit{is\ the\ final\ amount\ earned}$
- $P: \mathit{is\ the\ Principal\ (the\ initial\ amount\ of\ money\ borrowed\ or\ saved)}$
- $r: \mathit{is\ the}\ \boldsymbol{Yearly}\ \mathit{Percentage\ Rate, expressed\ as\ a\ decimal}\ (25\% = 0.25)$
- n: is the number of times yearly interest is compounded per year
- $t: \mathit{is\ time, in\ years}$

Example 4: Find the interest earned if \$8000 is deposited into an account paying 8% compounded daily, for seven years.

Solution 4: $\boldsymbol{A = P\left(1 + \frac{r}{n}\right)^{n(t)}}$

We sub in for the information. Compounded Daily means 365 times.

$$\boldsymbol{A = 8000\left(1 + \frac{0.08}{365}\right)^{365(7)}} \quad \rightarrow \quad \boldsymbol{A = 8000(1.00022)^{2555}} \quad \rightarrow \quad \boldsymbol{A = \$14\,004.52}$$

- Interest Earned: $\boldsymbol{A - P = I \quad so \quad \$14\,004.52 - 8000 = \$6004.52}$

Example 5: What initial investment is needed to become a millionaire is 30 years if you receive 11% interest compound monthly.

Solution 5: We sub in for the information. Compounded Monthly means 12 times.

$$\boldsymbol{1\,000\,000 = P\left(1 + \frac{0.11}{12}\right)^{12(30)}} \quad \rightarrow \quad \boldsymbol{1\,000\,000 = P(1.0092)^{360}} \quad \rightarrow \quad \boldsymbol{P = \frac{\$1\,000\,000}{(1.0092)^{360}}}$$

$\boldsymbol{P = \$36\,999,25}$ **We need an initial investment of** $\boldsymbol{\$36\,999.25}$

Growth and Decay Scenario

- There are 2 equations, we can use either one, but k needs to be determined

$A = A_0(x)^{\frac{t}{T}}$	$A = A_0(e)^{kt}$
A – Final Amount A_0 – Initial Amount x – Growth or Decay Value **Examples:** Half-Life: $x = 1/2$ Increase by 10%: $x = 1.1$ Decrease by 10%: $x = 0.9$ t – Total time that item remains T – Time of Growth or Decay	A – Final Amount A_0 – Initial Amount e – Mathematical Constant ≈ 2.71828 k – Proportional Constant t - time We will not use this until we learn more about Logarithms

Example 6: The half-life of Herlaarium-239 is about $45\ 000$ years. How much of a 3-gram sample will remain after 2000 years?

Solution 6:

Equation 1: $A = A_0(x)^{\frac{t}{T}} \quad \rightarrow \quad A = 3\left(\frac{1}{2}\right)^{\frac{2000}{45\ 000}} \quad \rightarrow \quad A = 2.91$

How much remains: $2.91\ Grams$

Equation 2: We do not know k so we need to figure that first (we'll learn this in Log Section)

$$A = A_0(e)^{kt} \quad \rightarrow \quad 1.5 = 3(e)^{k(45\ 000)} \quad \rightarrow \quad 0.5 = e^{45000k} \quad \rightarrow \quad k = \frac{\ln 0.5}{45000}$$

Because of Half-Life
$1.5\ is\ half\ of\ 3$

Now we can use the equation:

$$A = A_0(e)^{kt} \quad \rightarrow \quad A = 3(e)^{\frac{\ln 0.5}{45000}(2000)} \quad \rightarrow \quad A = 3e^{45000k} \quad \rightarrow \quad \boldsymbol{A = 2.91}$$

Section 5.1 – Practice Problems

1. Simplify the following statements

a) $\dfrac{\left(3^{\frac{1}{5}}\right)^{10} \cdot \left(3^{-3}\right)}{9}$

b) $\dfrac{\left(-4x^2y^{-2}\right)^{-3}}{x^{-1}y^2}$

c) $\dfrac{125^{3x-1} \cdot 25^{1-2x}}{\left(\frac{1}{5}\right)^{2x-3}}$

d) $\dfrac{2x^4 \cdot 3^{5x} - 4x^3 \cdot 3^{5x}}{x^3 - 2x^2}$

e) $\left(4^{-x} \cdot 8^x\right)^2$

f) $\dfrac{2^x(2^x + 2^{-x}) - 2^x(2^x - 2^{-x})}{2^{-2}}$

2. Solve for x

a) $4^{x^2 - x} = 1$

b) $3^{x^2} = 9 \cdot 3^{-x}$

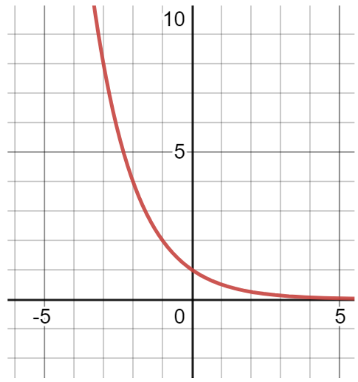
c) $4^{\sqrt{x + 1}} = 2^{3x - 2}$

d) $4^{-|x + 1|} = \frac{1}{16}$

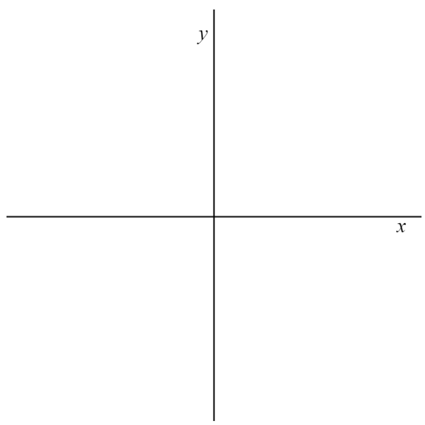
e) $4^{-2x + 1} = 8^{x - 4}$

f) $9^{2x - 1} = \left(\frac{1}{27}\right)^{x + 2}$

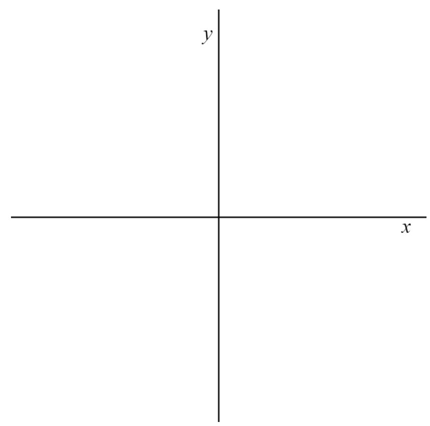
3. If $y = ab^x$ is defined by the graph below, what is the shape of the following:



a) $y = -ab^x$

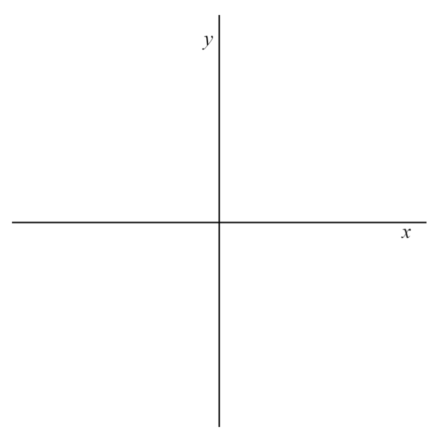


b) $y = ab^{-x}$

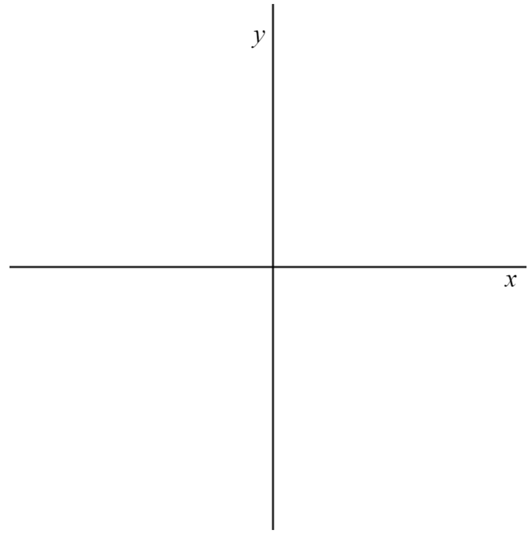


4. Explain the transformation listed in the equations below of the basic equation $y = 3^x$. Graph the transformation, identify the Domain, Range, intercepts(s), and asymptote(s).

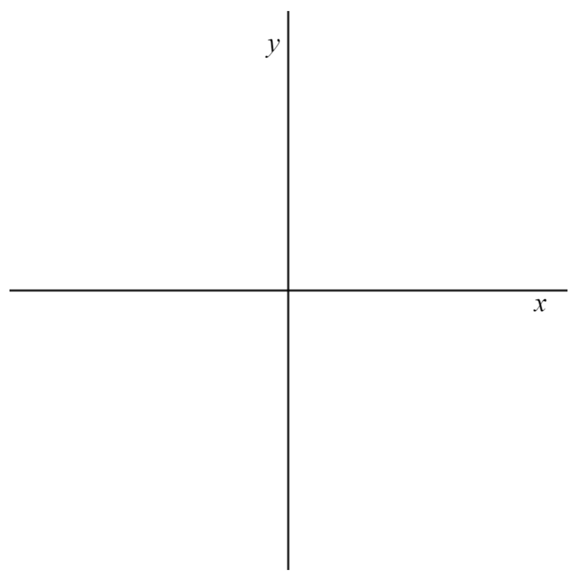
a) $y = 3^{x+2} - 3$



b) $y = \left(\frac{1}{3}\right)^{x} + 2$



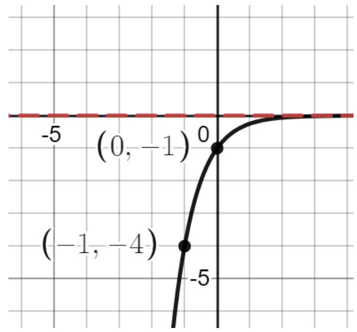
c) $y = -3^{-x}$



5. Match the Equation to the Graph

a) $y = 4^{-x}$	
b) $y = 3^{-x+1}$	
c) $y = 3^{-x} + 1$	
d) $y = -4^{-x}$	
e) $y = 2^{x+1} - 2$	
f) $y = -2^{x+2} + 1$	

A



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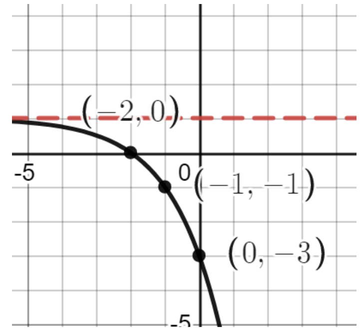

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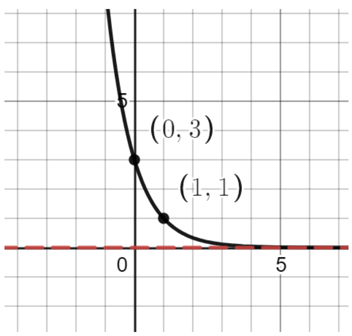

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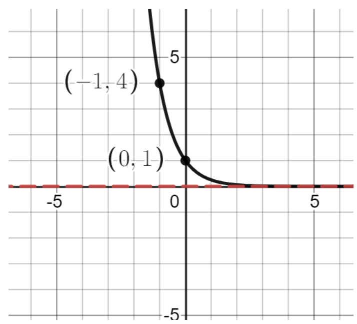
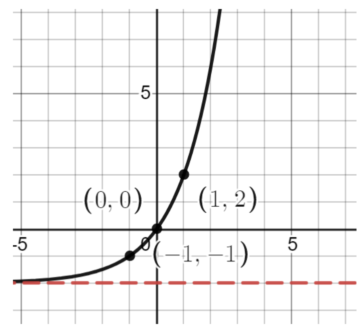
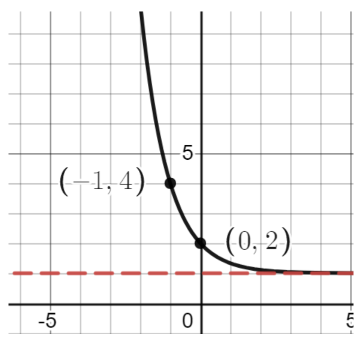


E



F



6. Find the base in the exponential function $y = b^x$ that contains the given point.

a) $(-1, 3)$

b) $(\frac{3}{2}, 27)$

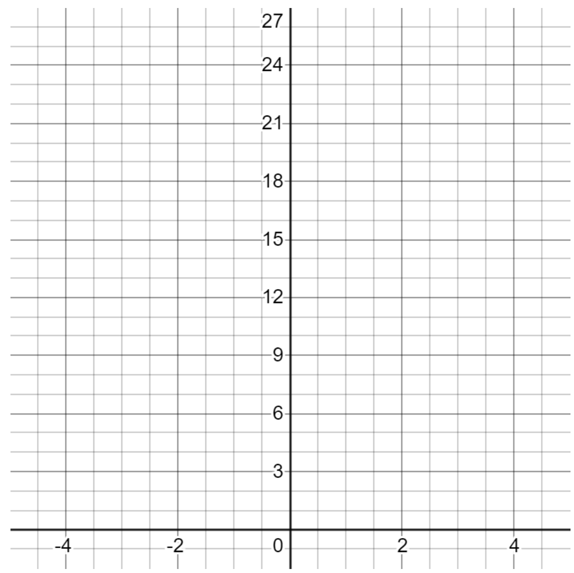
c) $(-\frac{2}{3}, \frac{1}{9})$

7. Find the exponential function in the form $c \cdot 2^{kx}$ that passes through $(0, 4)$ and $(12, 256)$.

8. Sketch the graph of $y = 2^x$ and $y = 3^x$ on the same grid.

x	-3	-2	-1	0	1	2	3
2^x							
3^x							

Discuss the behaviour of the two functions when $x < 0$ and $x > 0$.



9. Solve the given scenarios.

a) In 1876, an earthquake in Chile measured 8.9 on the Richter scale. How many times more powerful was this earthquake compared to the one in Utah that measured 6.4 on the Richter scale.
(The Richter scale is a power of ten scale)

b) If an earthquake in Alberta had an amplitude 1000 times larger than an earthquake that measured 4.9 on the Richter scale. What would the Alberta earthquake measure?

c) If you invest $\$1000$ at an annual interest rate of 6% compounded quarterly, what is the amount you would have in the account after $8\ years$? (No withdrawals made)

d) In the wasteland of the Commonwealth in Fallout 4, the safest locations to live are surrounded by radioactive isotopes with a half-life of $4\ minutes$. This means half the amount of the isotope decays every four minutes. If there was 84 grams of the isotope initially, how much is remaining after 23 minutes?

e) My brother managed to get me an investment account that pays 9.6% interest, compounded monthly. If I invested $\$12\,250$, how much money would I have after 10 years.

f) If the population of Sweden is around $30\,000\,000$ people in the year 2040, and the population continues to grow at 1.9% compounded yearly, what will the population be in the year 2072?

See Website for Detailed Answer Key

Extra Work Space