Section 4.4 – Graphing Rational Functions

- There are some particular steps to use in graphing a Rational Function

Recall that a Rational Function is of the form: $f(x) = \frac{g(x)}{h(x)} \quad g(x) \neq 0, h(x) \neq 0$

Step 1: Find **Vertical Asymptotes** by determining when the **denominator is zero**

Step 2: Find **Horizontal Asymptotes** by **dividing all the terms by the highest power of x** then **determine** the solution **when $x \to \infty$.**

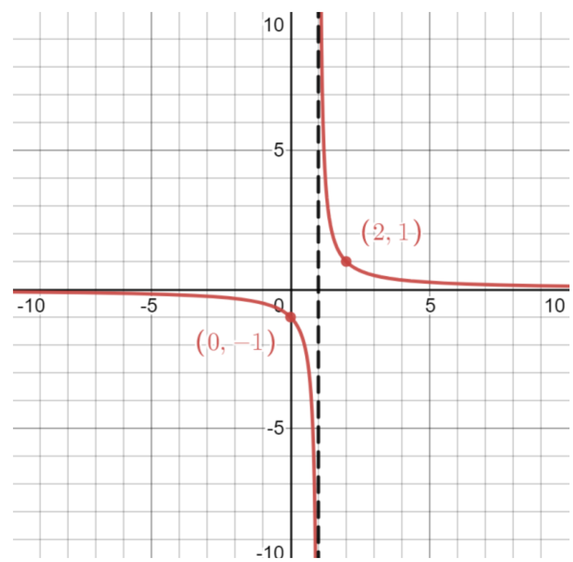
Step 3: Check to ensure there are **no Holes**; **factor the denominator** and see if any cancel

Step 4: Find the $\boldsymbol{x - intercept(s)}$ **and** $\boldsymbol{y - intercept}$, if they exist

Step 5: Make a **table of values** to determine the general behaviour and shape. **Check behaviour** specifically on **either side of the vertical asymptote(s)**

Step 6: Draw a **smooth curve** through the given points

Example 1: Graph the Rational function $g(x) = \frac{1}{x-1}$



Solution 1: **Vertical asymptote is: $x = 1$**

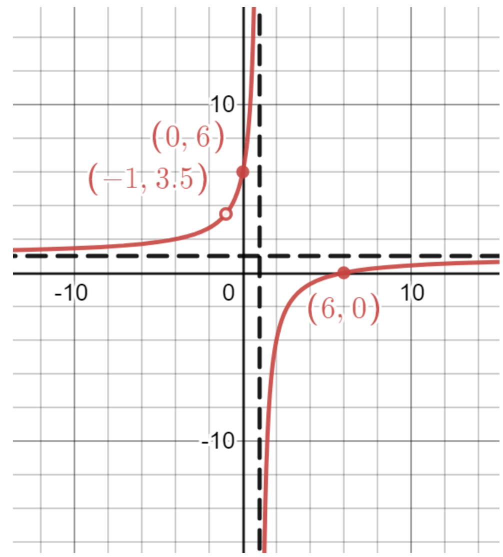
Horizontal asymptote is: $y = 0$

$$\frac{1}{x-1} = \frac{\frac{1}{x}}{\frac{x}{x} - \frac{1}{x}} \to \frac{\frac{1}{\infty}}{1 - \frac{1}{\infty}} = \frac{0}{1} = 0$$

x	0	0.9	1.1	-10	10
$g(x)$	-1	-10	10	-0.1	0.1

x	0.9999	1.0001	10000	-10000	2
$g(x)$	-10000	10000	0.0001	-0.0001	1

Example 2: Graph the Rational function $f(x) = \frac{x^2 - 5x - 6}{x^2 - 1}$
With all the pertinent info. $x - intercept(s)$,
$y - intercept$, asymptote(s),
hole (if applicable)



Solution 2:

$$f(x) = \frac{x^2 - 5x - 6}{x^2 - 1} \rightarrow \frac{(x-6)(x+1)}{(x-1)(x+1)} \rightarrow \frac{(x-6)}{(x-1)}$$

Vertical asymptote is: $\boldsymbol{x = 1}$

Hole at: $\boldsymbol{x = -1}$

$$f(-1) = \frac{(-1-6)}{(-1-1)} = \frac{-7}{-2} = \frac{7}{2}$$

Hole at:

$$\left(\boldsymbol{-1, \frac{7}{2}}\right)$$

Horizontal asymptote is: $\boldsymbol{y = 1}$

$$\frac{x^2 - 5x - 6}{x^2 - 1} \rightarrow \frac{\frac{x^2}{x^2} - \frac{5x}{x^2} - \frac{6}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}} \rightarrow \frac{1 - \frac{5}{\infty} - \frac{6}{\infty^2}}{1 - \frac{1}{\infty^2}} \rightarrow \frac{1 - 0 - 0}{1 - 0} \rightarrow \frac{1}{1} = \mathbf{1}$$

$\boldsymbol{y - intercept}$

Set $\boldsymbol{x = 0}$,

$$f(0) = \frac{(0)^2 - 5(0) - 6}{(0)^2 - 1}$$

$$\boldsymbol{f(0) = \frac{-6}{-1} = 6}$$

$$\boldsymbol{y - intercept\text{:}\ (0, 6)}$$

$\boldsymbol{x - intercept}$

Set $\boldsymbol{y = 0}$, we only care about the numerator

$$0 = x^2 - 5x - 6$$

$$0 = (x-6)(x+1)$$

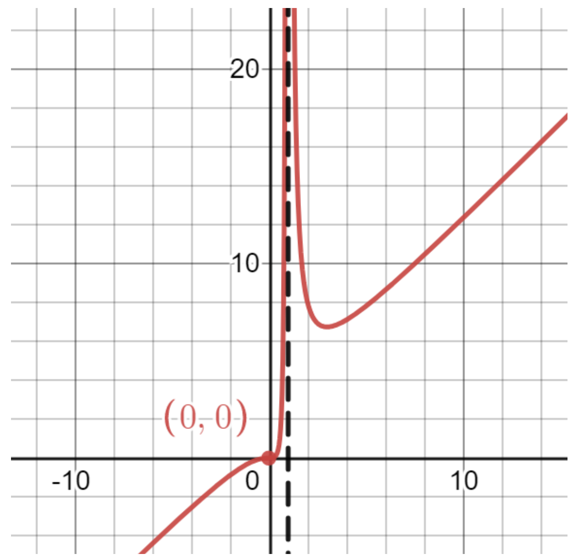
Hole, so Discard as a root

$$\boldsymbol{0 = (x - 6)}$$

$$\boldsymbol{x - intercept(s)\text{:}\ (6, 0)}$$

x	0	0.9	1.1	-10	10	1000	-1000
$f(x)$	6	51	-49	1.45	0.444	0.995	1.005

Example 3: Graph the Rational function $f(x) = \dfrac{x^3}{x^2 - 2x + 1}$
With all the pertinent info. $x - intercept(s)$, $y - intercept$, asymptote(s), hole (if applicable)



Solution 3:

$$f(x) = \frac{x^3}{x^2 - 2x + 1} \quad \rightarrow \quad \frac{x^3}{(x-1)(x-1)}$$

Vertical asymptote is: $\boldsymbol{x = 1}$

No Holes

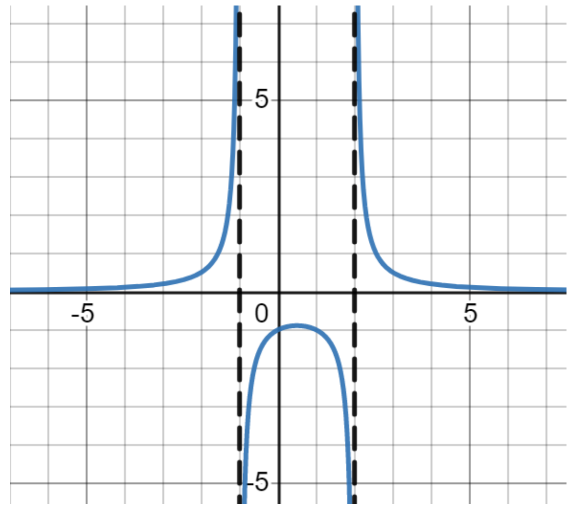
Horizontal asymptote is: $\boldsymbol{None}$

$$\frac{x^3}{x^2 - 2x + 1} \rightarrow \frac{\frac{x^3}{x^2}}{\frac{x^2}{x^3} - \frac{2x}{x^3} + \frac{1}{x^3}} \rightarrow \frac{\infty}{\frac{1}{\infty} - \frac{3}{\infty^2} + \frac{1}{\infty^3}} \rightarrow \frac{\infty}{0 - 0 - 0} \rightarrow \frac{\infty}{0} = \boldsymbol{Undefined}$$

$\boldsymbol{y - intercept}$	$\boldsymbol{x - intercept}$
Set $\boldsymbol{x = 0}$,	**Set $\boldsymbol{y = 0}$, we only care about the numerator**
$f(0) = \dfrac{(0)^3}{(0-1)(0-1)}$	$0 = x^3$ $0 = x$
$\boldsymbol{f(0) = \frac{0}{1} = 0}$	$\boldsymbol{0 = x}$
$\boldsymbol{y - intercept}$: $\boldsymbol{(0, 0)}$	$\boldsymbol{x - intercept(s)}$: $\boldsymbol{(0, 0)}$

x	0	0.9	1.1	−10	10	1000	−1000
$f(x)$	0	72.9	133.1	−8.26	12.35	1002	−998

Example 4: Graph the Rational function $f(x) = \frac{2}{x^2 - x - 2}$
With all the pertinent info. $x - intercept(s)$,
$y - intercept$, asymptote(s),
hole (if applicable)



Solution 4:

$$f(x) = \frac{2}{x^2 - x - 2} \quad \rightarrow \quad \frac{2}{(x-2)(x+1)}$$

Vertical asymptote is: $x = -1, 2$

No Holes

Horizontal asymptote is: $y = 0$

$$\frac{2}{x^2 - x - 2} \rightarrow \frac{\frac{2}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}} \rightarrow \frac{\frac{2}{\infty^2}}{1 - \frac{1}{\infty} - \frac{2}{\infty^2}} \rightarrow \frac{0}{1 - 0 - 0} \rightarrow \frac{0}{1} = \mathbf{0}$$

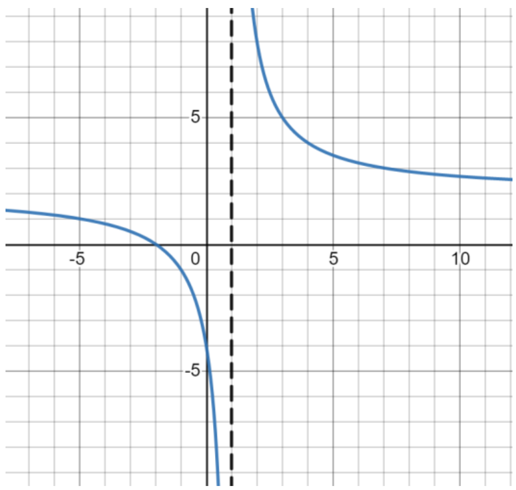
$y - intercept$	$x - intercept$
Set $x = 0$, $f(0) = \frac{2}{(0)^2 - (0) - 2}$ $\boldsymbol{f(0) = \frac{2}{-2} = -1}$ $\boldsymbol{y - intercept: (0, -1)}$	**Set $y = 0$, we only care about the numerator** $0 = 2$ $\boldsymbol{x - intercept(s): None}$

x	0	-0.9	-1.1	1.9	2.1	10	-10
$f(x)$	-1	-6.9	6.5	-6.9	6.5	0.02	0.02

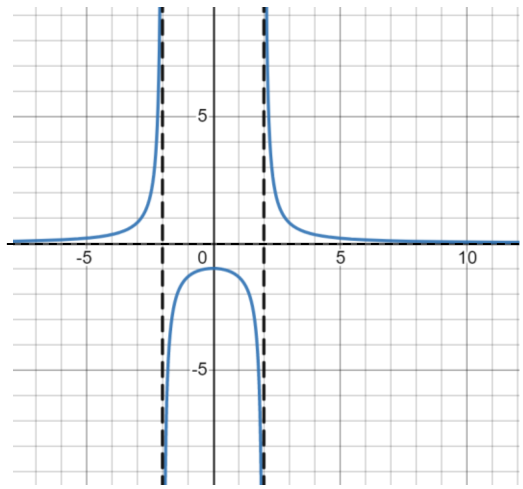
Section 4.4 – Practice Problems

1. For the following functions, find the Domain, the Vertical and Horizontal Asymptotes (if any), and approximate any $x - intercept(s)$ and $y - intercept(s)$

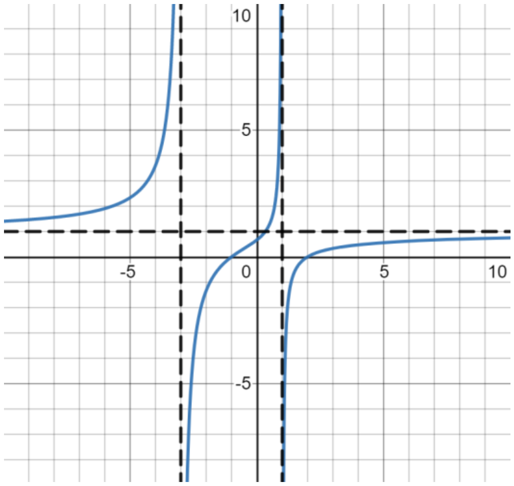
a) $f(x) = \dfrac{2x+4}{x-1}$



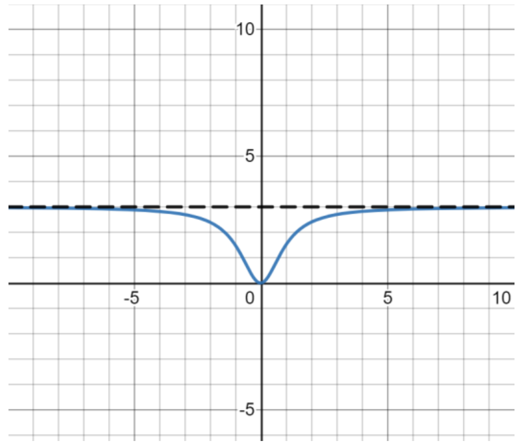
b) $f(x) = \dfrac{4}{x^2-4}$



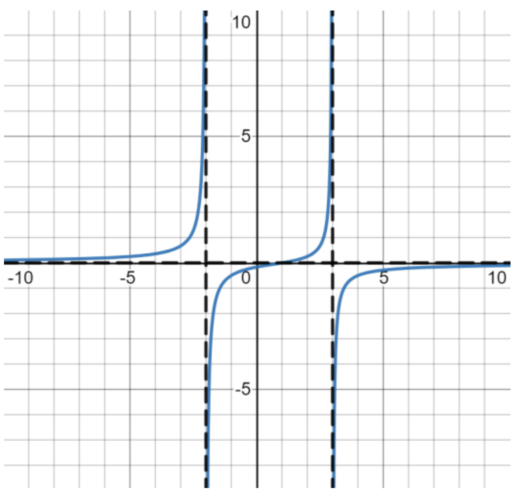
c) $f(x) = \dfrac{x^2 - x - 2}{x^2 + 2x - 3}$



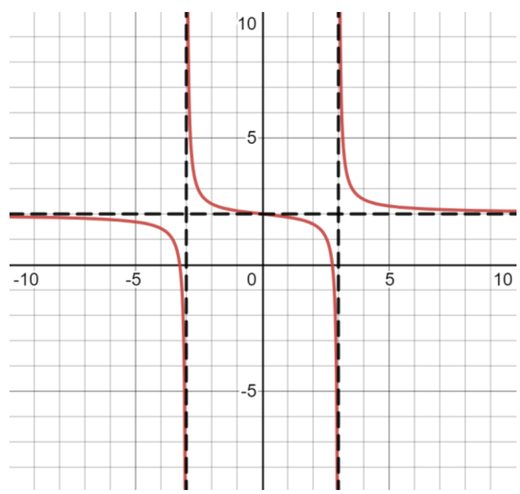
d) $f(x) = \dfrac{3x^2}{x^2 + 1}$



e) $f(x) = \dfrac{1 - x}{x^2 - x - 6}$

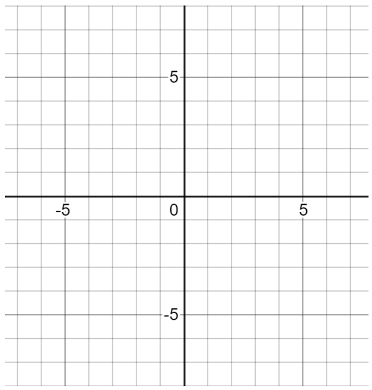


f) $f(x) = \dfrac{x}{x^2 - 9} + 2$

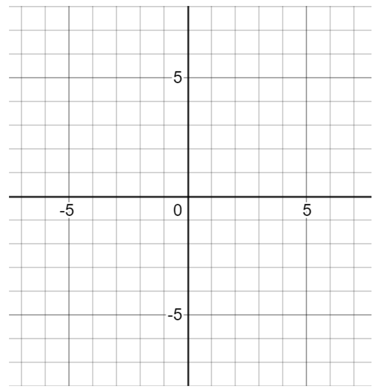


2. Find the Hole in the following Functions, sketch the graph and show where the Hole appears.

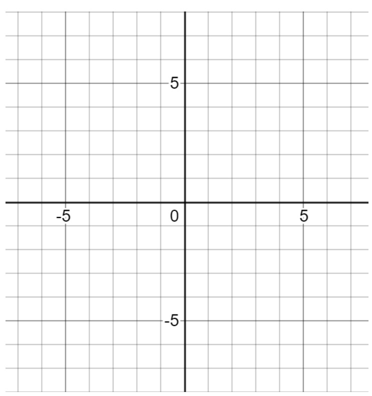
a) $f(x) = \dfrac{x^2 - 4}{x - 2}$



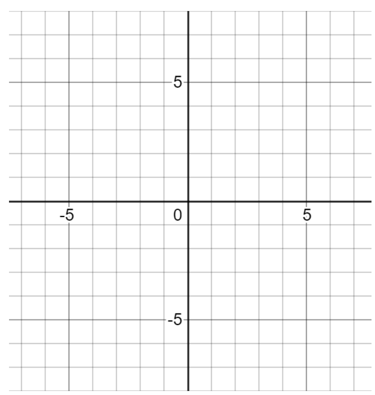
b) $f(x) = \dfrac{x^2 - 1}{x + 1}$



c) $y = \dfrac{x^2 - 9}{3 - x}$



d) $y = \dfrac{4 - x^2}{x + 2}$



3. Fill in the table of values for the following functions. Get comfortable with calculators and observe the behaviour of functions as they approach asymptotes.

a) $f(x) = \dfrac{3}{x-1}$

x	0.5	1.5	0.9	1.1	0.99	1.01
$f(x)$						

x	10	100	1000	-10	-100	-1000
$f(x)$						

b) $f(x) = \dfrac{3x^2-1}{x^2}$

x	-0.5	0.5	-0.1	0.1	-0.01	0.01
$f(x)$						

x	10	100	1000	-10	-100	-1000
$f(x)$						

c) $f(x) = \dfrac{x}{x-2}$

x	1.5	2.5	1.9	2.1	1.99	2.01
$f(x)$						

x	10	100	1000	-10	-100	-1000
$f(x)$						

4. Find the roots (zero's, solution, $x - intercepts$), if they exist, of the Rational Functions

a) $f(x) = \dfrac{x^2 - 4}{x + 2}$

b) $g(x) = 1 - \dfrac{3}{x^2 + 2}$

c) $h(x) = 1 - \dfrac{3}{x - 3}$

d) $f(x) = -1 + \dfrac{4}{x^2 + 1}$

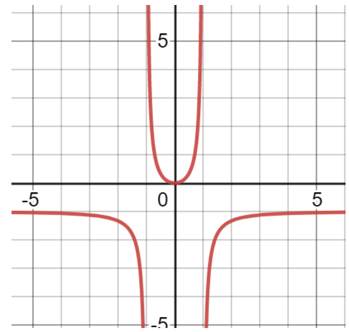
e) $g(x) = 1 + \dfrac{4}{x^2 + 1}$

f) $h(x) = \dfrac{x^3 + 8}{x^2 + 4}$

5. Match the equation with the graph.

a) $h(x) = \dfrac{1}{x-1}$		b) $h(x) = \dfrac{x}{x-1}$		c) $h(x) = \dfrac{-2}{x-1}$	
d) $h(x) = \dfrac{1}{x^2}$		e) $h(x) = \dfrac{-x^2}{x^2-1}$		f) $h(x) = \dfrac{x^2}{x^2-1}$	
g) $h(x) = \dfrac{x-3}{x-1}$		h) $h(x) = \dfrac{4}{x^2+1}$		i) $h(x) = \dfrac{x}{x^2-4}$	
j) $h(x) = \dfrac{x^2-2x}{x^2+2x+1}$					

A



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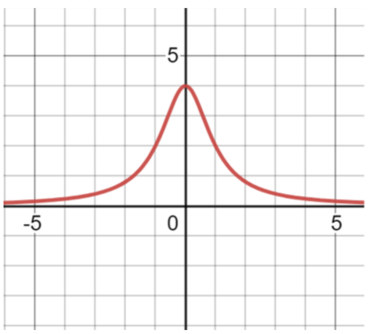

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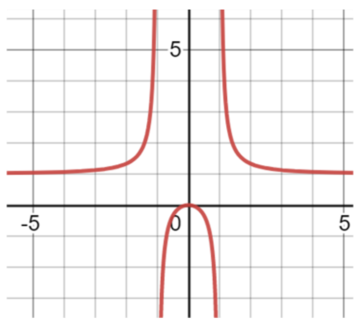

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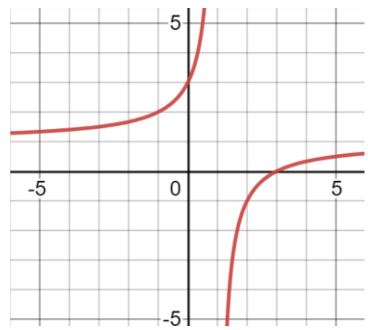

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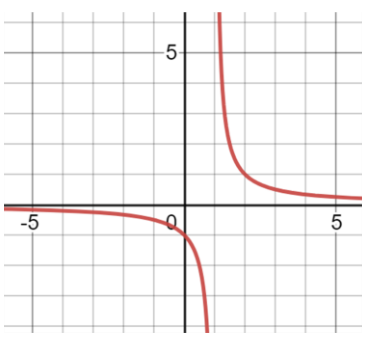

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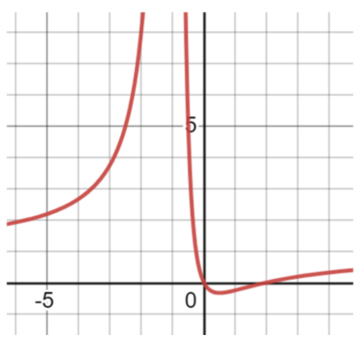

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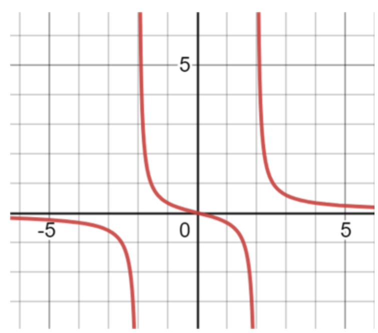

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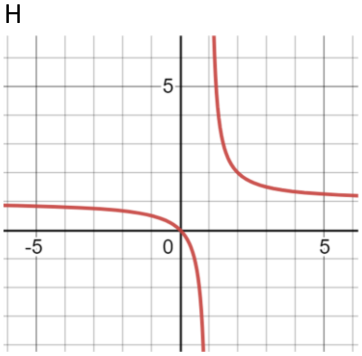

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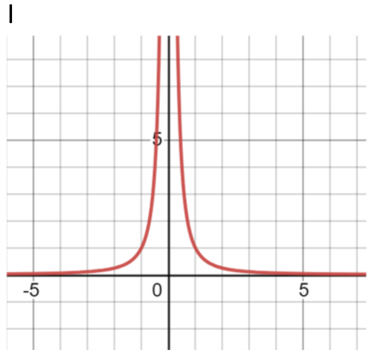

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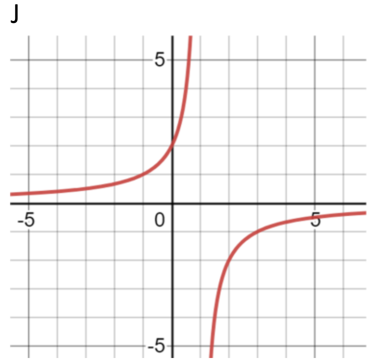
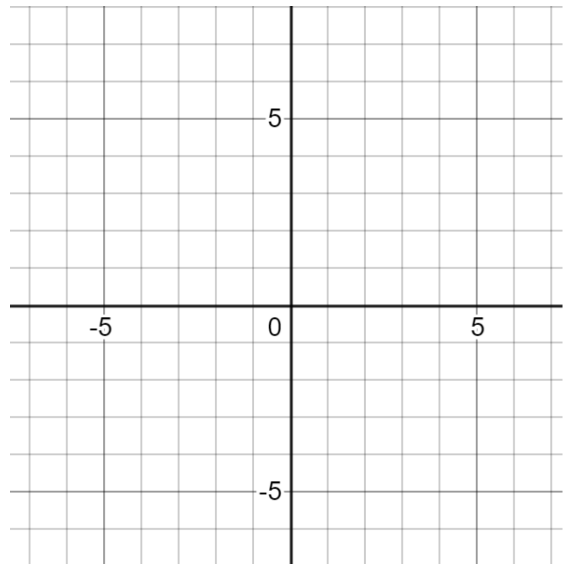


6. Sketch the following Rational Functions. State the Domain, the $x-intercepts$ and $y-intercepts$, identify the vertical asymptotes, horizontal asymptotes, and holes. Plot additional points to help generate the graph.

a) $h(x)=\dfrac{-x}{x+2}$



x						
$f(x)$						

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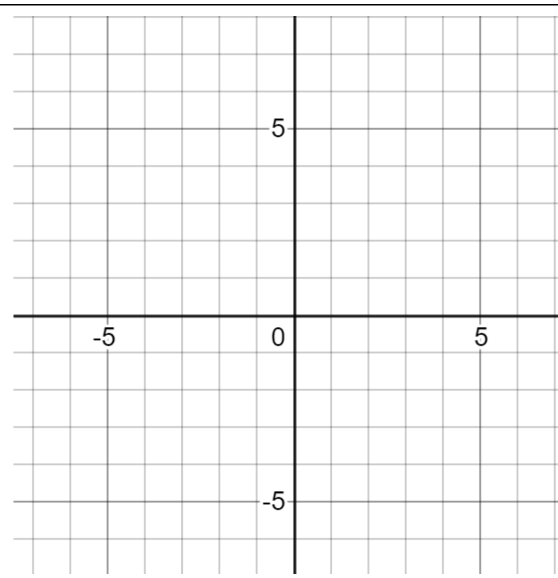
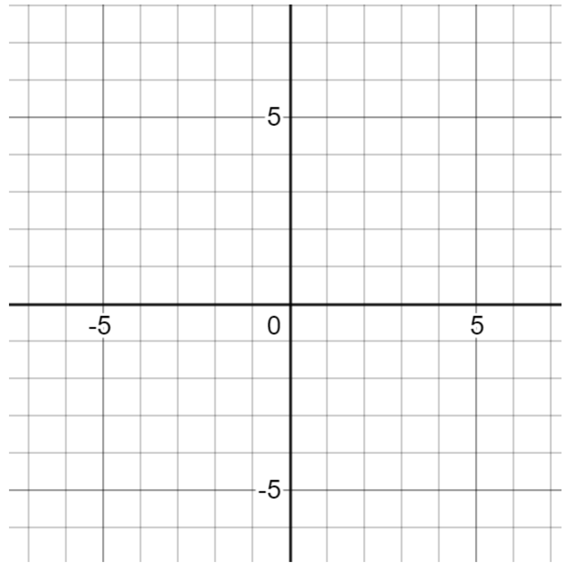
b) $h(x)=\dfrac{x+2}{x-1}$



x						
$f(x)$						

c) $h(x) = \dfrac{x^2 + 3x + 2}{x^2 - 4}$



x						
$f(x)$						

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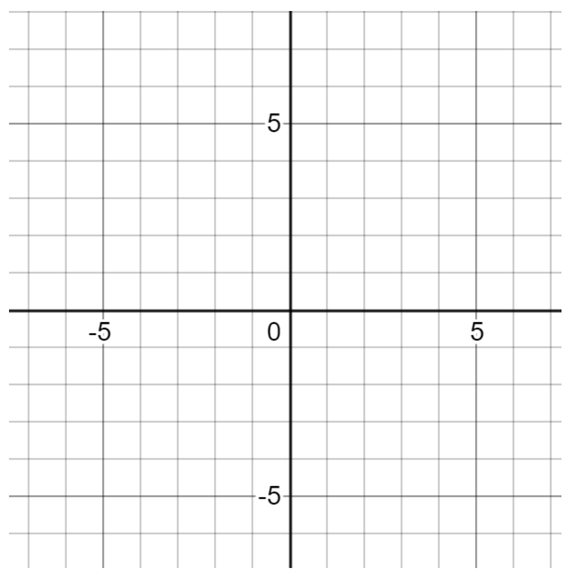
d) $h(x) = \dfrac{x^2}{x^3 - 9x}$



x						
$f(x)$						

See Website for Detailed Answer Key of the Remainder of the Questions

Extra Work Space