Section 4.4 – Practice Problems

1. Find two numbers whose difference is 150 and whose product is a minimum.

Let x and y be the numbers

$x - y = 150$

$xy = \min$

$x(x-150) = P$

$x^2 - 150x = P$

↑ parabola with min

$y = x - 150$

$P' = 2x - 150$

$0 = 2x - 150$

$\boxed{x = 75}$

$y = 75 - 150$

$\boxed{y = -75}$

P has min at $x = 75$, $y = -75$

2. Find two positive numbers with product 200 such that the sum of one number and twice the second number is as small as possible.

Let $xy = 200$ $y = \frac{200}{x}$

$x + 2y = \min$

$x + 2\left(\frac{200}{x}\right) = S$

$S' = 1 - \frac{400}{x^2}$

$0 = 1 - \frac{400}{x^2}$

$x = \pm 20$

min

$-$ | $+$ at 20

if $x = 20$ $y = 10$

$x = -20$ $y = -10$

positive answers only.

$\boxed{x = 20 \quad y = 10}$

3. A rectangle has a perimeter of 100cm. What length and width should it have so that its area is a maximum.

$2l + 2w = 100$

$l + w = 50$

$l = 50 - w$

$l \cdot w = A$

$(50 - w)w = A$

$50w - w^2 = A$

↑ open down parabola has max

$A' = 50 - 2w$

$\boxed{w = 25 \quad l = 25}$

4. Show that a rectangle with given area has minimum perimeter where it is a square.

$A = LW \rightarrow L = \frac{A}{W}$

$P = 2L + 2W$

$P = 2\left(\frac{A}{w}\right) + 2w \rightarrow P = \frac{2A}{w} + 2w$

$P' = \frac{-2A}{w^2} + 2$

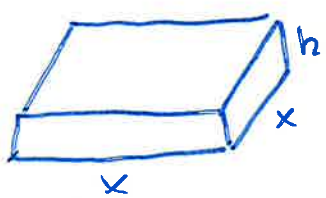
$P' = \frac{2w^2 - 2A}{w^2}$

$0 = 2w^2 - 2A$

min when $A = w^2$

therefore it is a square because l must equal w.

5. A box with a square base and open top must have a volume of 4000cm^3. Find the dimensions of the box that minimizes the amount of material used.



$V = x^2h$

$4000 = x^2h$

$\frac{4000}{x^2} = h$

$SA = x^2 + 4xh$

$x^2 + 4x\left(\frac{4000}{x^2}\right)$

$= x^2 + \frac{16000}{x}$

$SA' = 2x - \frac{16000}{x^2}$

$SA' = \frac{2x^3 - 16000}{x^2}$

$0 = 2x^3 - 16000$

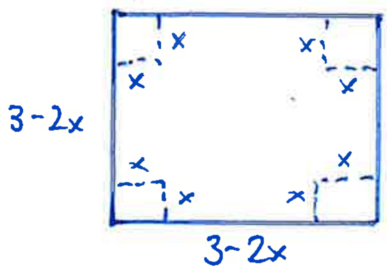
$\frac{16000}{2} = x^3$

$8000 = x^3$

$20 = x$

$\boxed{\text{if } x = 20 \quad h = 10}$

6. A box with an open top is to be constructed from a square piece of cardboard, 3m wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that the box can have.



$V = (3-2x)^2 x$

$V' = 2(3-2x)(-2)x + (3-2x)^2$

$= -4x(3-2x) + (3-2x)^2$

$= (3-2x)[-4x + 3 - 2x]$

$= (3-2x)(-6x+3)$

$= (3-2x)(3)(1-2x)$

$0 = (3-2x)(1-2x)$

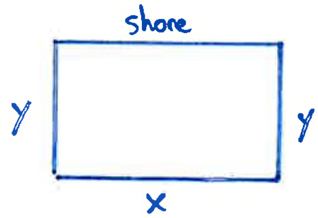
reject $\rightarrow$ $x = \frac{3}{2}$ gives $V=0$ $\quad x = \frac{1}{2}$

$V(\frac{3}{2}) = 0$

$\boxed{V(\frac{1}{2}) = 2m^3}$

7. A lifeguard at a public beach has 400m of rope available to lay out a rectangular restricted swimming area using the straight shoreline as one side of the rectangle.

a) If she wants to maximize the swimming area, what will the dimensions of the rectangle be?



$P = 2y + x$

$400 = 2y + x$

$x = 400 - 2y$

$xy = A$

$(400-2y)y = A \qquad A = 400y - 2y^2$

$A' = 400 - 4y \rightarrow 0 = 100 - y$

$y = 100$

$\boxed{y = 100m \quad x = 200m}$

b) To ensure the safety of the swimmers, she decides that nobody should swim more than 50m from shore. What should the dimensions of the swimming area be with the added restriction.

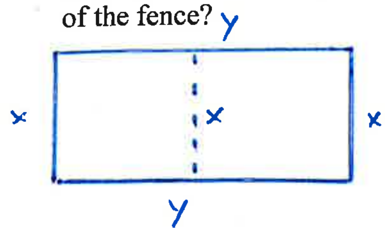
$A = 400y - 2y^2$ but $0 < y \leq 50$

no crit pts in this interval

so $y = 50$

$\boxed{y = 50m \quad x = 300m}$

8. A farmer wants to fence an area of 750 000m² in a rectangular field and divide it in half with a fence parallel to one of the sides of the rectangle. How can this be done so as to minimize the cost of the fence?



$P = 2y + 3x \qquad A = xy \qquad 750\,000 = xy$

$\frac{750\,000}{x} = y$

$P = 2\left(\frac{750\,000}{x}\right) + 3x$

$P = \frac{1\,500\,000}{x} + 3x$

$P' = 3 - \frac{1\,500\,000}{x^2}$

$0 = 3 - \frac{1\,500\,000}{x^2}$

$\frac{1500000}{x^2} = 3$

$1500000 = 3x^2$

$500000 = x^2$

$$\boxed{\begin{array}{l} x \doteq 707\text{m} \\ y \doteq 1061\text{m} \end{array}}$$

9. Show that if the function $y = f(x)$ has a minimum at c, then the function $y = \sqrt{f(x)}$ also has a minimum at c.

$y' = $ ^ a

$f'(c)$ 0

for

$x < c \quad f(x) < f(c)$

$x > c \quad f(x) > f(c)$

$\sqrt{f(x)}$ must be > 0

then $f'(x) < 0$

for $x < c$

and $f'(x) > 0$

$x > c$

so it has a minimum

$y' = \frac{1\,f'(x)}{2\sqrt{f(x)}}$

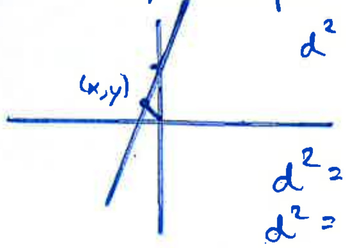
$0 = \frac{1\,f'(x)}{2\sqrt{f(x)}}$

occurs at $f'(x) = 0$

which is at $f'(c)$

10. Find the point on the line $y = 5x + 4$ that is closest to the origin.

say some point (x,y) is closest to the origin



$d^2 = (y-0)^2 + (x-0)^2 \qquad y = 5x + 4$

$d^2 = y^2 + x^2$

$d^2 = (5x+4)^2 + x^2$

$d^2 = 25x^2 + 40x + 16 + x^2$

$d^2 = 26x^2 + 40x + 16$

$(d^2)' = 52x + 40$

$0 = 52x + 40$

$x = -\frac{40}{52} = -\frac{10}{13}$

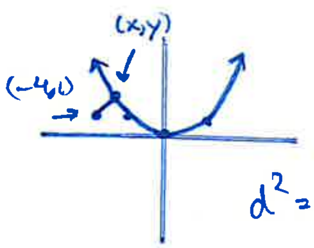
min

$-$ | $+$

$-\frac{10}{13}$

$$\boxed{f\left(-\tfrac{10}{13}\right) = \tfrac{2}{13}}$$

11. Find the point on the parabola $2y = x^2$ that is closest to the point $(-4, 1)$.



$y = \frac{1}{2}x^2 \qquad y = \frac{1}{2}x^2$

$d^2 = (y-1)^2 + (x+4)^2$

$d^2 = \left(\frac{1}{2}x^2 - 1\right)^2 + (x+4)^2$

$d^2 = \frac{1}{4}x^4 - x^2 + 1 + x^2 + 8x + 16$

$d^2 = \frac{1}{4}x^4 + 8x + 17 \rightarrow (d^2)' = x^3 + 8$

no factorable

$(d^2)' = (x+2)(x^2 - 2x + 4)$

$0 = \qquad x = -2$

min

$-$ | $+$

-2

$y = \frac{1}{2}(-2)^2$

$y = 2$

$$\boxed{(-2, 2)}$$

12. A can is to be made to hold a litre of oil. Find the radius of the can that will minimize the cost of the metal to make the can. ($1L = 1000cm^3$).

$SA = 2\pi r^2 + 2\pi rh \qquad V = \pi r^2 h \qquad 1000 = \pi r^2 h \qquad h = \frac{1000}{\pi r^2}$

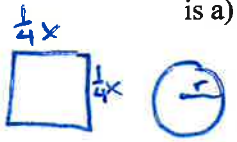
$2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \qquad SA' = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2}$

$2\pi r^2 + \frac{2000}{r} \qquad 0 = 4\pi r^3 - 2000 \qquad \boxed{r = \sqrt[3]{\frac{500}{\pi}}}$

$0 = r^3 - \frac{500}{\pi}$

Min: $- \mid +$ at $\sqrt{\frac{500}{\pi}}$

13. A piece of wire 40cm long is cut into two pieces. One piece is bent into the shape of a square and the other is bent into the shape of a circle. How should the wire be cut so that the total area enclosed is a) a maximum and b) a minimum.



$A_1 = x^2$

$A_2 = \pi r^2$

$x + 2\pi r = 40$

$r = \frac{40 - x}{2\pi}$

Total Area $= \left(\frac{1}{4}x\right)^2 + \pi r^2$

$= \left(\frac{1}{4}x\right)^2 + \pi\left(\frac{40-x}{2\pi}\right)^2$

$\Rightarrow \frac{1}{16}x^2 + \frac{(40-x)^2}{4\pi}$

$\frac{1}{16}x^2 + \frac{1600 - 80x + x^2}{4\pi}$

$\Rightarrow A' = \frac{x}{8} + \frac{1}{4\pi}(-80 + 2x)$

$= \frac{x}{8} - \frac{80}{4\pi} + \frac{2x}{4\pi}$

$0 = \frac{x}{8} - \frac{20}{\pi} + \frac{x}{2\pi}$

$0 = \frac{\pi x}{4} - 40 + x$

$40 = \frac{\pi x}{4} + x$

$40 = x\left(\frac{\pi}{4} + 1\right)$

$\frac{40}{\left(\frac{\pi}{4} + 1\right)} = x \qquad x = 22.4$

local min: $- \mid +$ at 22.4

Endpoints

$A(0) = 127 cm^2$

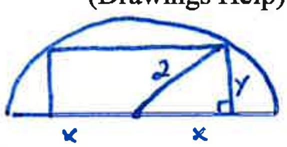
$A(40) = 100 cm^2$

$A(22.4) = 56 cm^2$

max: use all for circle

min: 22.4 for sq, 17.6 for cir

14. A rectangle is inscribed in a semicircle of radius 2cm. Find the largest area of such a rectangle. (Drawings Help)



$A = 2xy$

$\boxed{\text{Equation of circle with radius 2} \quad x^2 + y^2 = 4}$

$y = \sqrt{4 - x^2}$

$A = 2x\sqrt{4 - x^2}$

$A' = 2x \cdot \frac{1(-2x)}{2\sqrt{4-x^2}} + \sqrt{4 - x^2} \cdot 2$

$\frac{-2x^2}{\sqrt{4-x^2}} + \frac{2(4-x^2)}{\sqrt{4-x^2}} = \frac{-2x^2 + 8 - 2x^2}{\sqrt{4-x^2}} = \frac{-4x^2 + 8}{\sqrt{4-x^2}}$

$0 = -4x^2 + 8$

$2 = x^2$

$\sqrt{2} = x$

$+ \mid -$ at $\sqrt{2}$ max

$A = 2\sqrt{2}\sqrt{2}$

$= \boxed{4 cm^2}$

15. Solve the problem in Example 4 if it costs \$120/m to lay the cable underwater.

Total Cost: $C(x) = 40x + 120\sqrt{(1200-x)^2 + 100^2}$

$C'(x) = 40 + \frac{60 \cdot 2(1200-x)(-1)}{\sqrt{(1200-x)^2 + 100^2}} \rightarrow 40 - \frac{120(1200-x)}{\sqrt{(1200-x)^2 + 100^2}} = 0$

$40\sqrt{(1200-x)^2 + 100^2} - 120(1200 - x) = 0$

$\sqrt{(1200-x)^2 + 100^2} - 3(1200 - x) = 0$

$\sqrt{(1200-x)^2 + 100^2} = -3x + 3600$

$(1200 - x)^2 + 100^2 = 9x^2 - 21600x + 12\,960\,000$

$1440\,000 - 2400x + x^2 + 10\,000 = 9x^2 - 21600x + 12\,960\,000$

$8x^2 - 19200x + 11\,510\,000 = 0$ use quad eq^n or Desmos.

$x \doteq 1165$

$x \doteq 1235 \leftarrow$ outside Domain

$C(0) = \$144\,499$

$C(1200) = \$60\,000$

$\boxed{C(1165) = \$59314}$ min