4.4 Applied Maximum and Minimum Problems

A very important application of derivatives occurs in the solution of 'optimization' problems, where we must maximize or minimize quantities.

Like our Section 3 Applications questions, there are steps to consider to simplify the process.

1. **Understand the Problem.** Read the problem carefully and ask: What do I know? What do I need to know?
2. **Draw a Diagram.** A diagram is a useful tool when working through these scenarios
3. **Introduce Notation.** Assign variables to the quantities in question.
4. Express multi-variable equation in terms of **one variable** when possible.
5. Consider the **Domain** of the Function.

Ex 1: Find two positive numbers whose product is 10 000 and whose sum is a minimum.

Let x be the first number and y be the second number

$S = x + y$

$xy = 10000$

$y = \frac{10\,000}{x}$

write in terms of one variable

$S = x + \frac{10\,000}{x}$, $x > 0$

$\frac{dS}{dx} = 1 - \frac{10\,000}{x^2} = \frac{x^2 - 10\,000}{x^2}$

$\frac{x^2 - 10\,000}{x^2} = 0$

$x^2 - 10000 = 0$

$x^2 = 10\,000$

$x = \pm 100$ but $x > 0$

$x = 100$

critical numbers: 100, 0 (but $x > 0$)

this is our critical number

$\frac{dS}{dx} > 0$ when $x > 100$

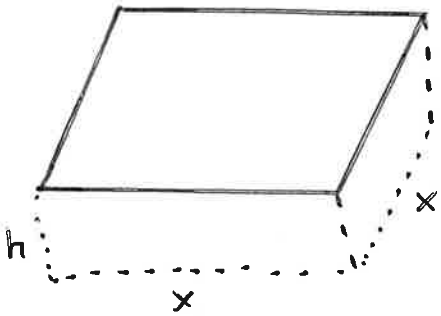
$\frac{dS}{dx} < 0$ when $0 < x < 100$

so minimum at $x = 100$

$y = \frac{10\,000}{100} = 100$

$\boxed{x = 100 \quad y = 100}$

Ex 2: If 2700 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of the box.



$V = x^2 h$

$A = 4xh + x^2$

$2700 = 4xh + x^2$

$2700 - x^2 = 4xh$

$\frac{2700 - x^2}{4x} = h$

$V = x^2\left(\frac{2700 - x^2}{4x}\right)$

$V = \frac{2700x - x^3}{4}$ $\rightarrow$ $675x - \frac{1}{4}x^3$

$\frac{dV}{dx} = 675 - \frac{3}{4}x^2$

Domain Restrictions

$2700 - x^2 \geq h$

but $x, h \geq 0$

$2700 - x^2 \geq 0$

$2700 \geq x^2$

$\sqrt{2700} \geq x$

$\sqrt{9}\sqrt{3}\sqrt{100} \geq x$

$\boxed{x \leq 30\sqrt{3}}$

Critical Numbers:

$\frac{dV}{dx} = 0$

$0 = 675 - \frac{3}{4}x^2$

$\frac{3}{4}x^2 = 675$

$x^2 = 900$

$\boxed{x = 30}$

Absolute Max on interval $[0, 30\sqrt{3}]$ and crit point $x = 30$

$V(30) = 675(30) - \frac{1}{4}(30)^3 = 13\,500$

$V(0) = 0$

$V(30\sqrt{3}) = 675(30\sqrt{3}) - \frac{1}{4}(30\sqrt{3})^3 = 0$

Absolute Max: $x = 30$

$\boxed{V = 13\,500 cm^3}$

Ex 3: Find the points on parabola $y = 6 - x^2$ that are closest to the point $(0, 3)$.

$d^2 = x^2 + y^2$

$y = 6 - x^2$

$d = \sqrt{x^2 + y^2}$

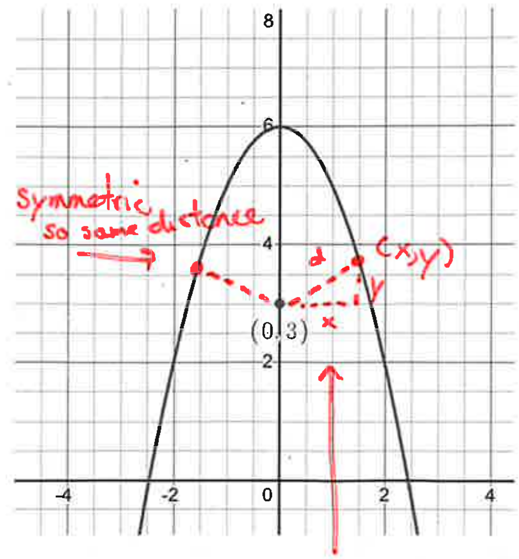
$d = \sqrt{x^2 + (y-3)^2}$

$d = \sqrt{x^2 + (6 - x^2 - 3)^2}$

$d = \sqrt{x^2 + (3 - x^2)^2}$

$d = \sqrt{x^2 + 9 - 6x^2 + x^4}$

$d = \sqrt{x^4 - 5x^2 + 9}$



$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y-3}{x-0} = \frac{y-3}{x}$

$\downarrow$

$d^2 = x^4 - 5x^2 + 9$

minimizing d^2 also minimizes d so its easier to minimize d^2

$f(x) = x^4 - 5x^2 + 9$

$f'(x) = 4x^3 - 10x$

$f'(x) = 4x^3 - 10x = 0$

$2x(2x^2 - 5) = 0$

critical points: $x = 0$

$x = \pm\sqrt{\frac{5}{2}}$

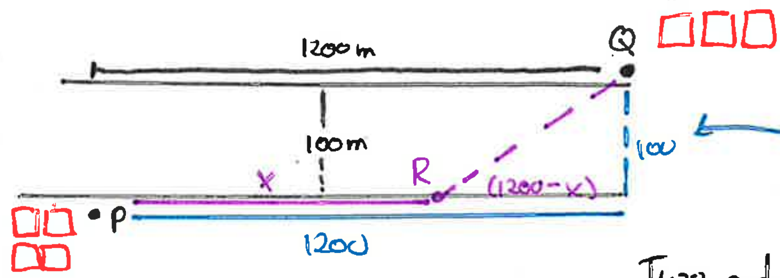
$y = 6 - 0^2 = 6$ ← Absolute Maximum.

$y = 6 - \left(\sqrt{\frac{5}{2}}\right)^2 = 3.5$

$y = 6 - \left(-\sqrt{\frac{5}{2}}\right)^2 = 3.5$

Points closest to parabola exist at $\left(\sqrt{5/2}, 3.5\right)$ and $\left(-\sqrt{5/2}, 3.5\right)$

Ex 4: A cable television company is laying cable in an area with underground utilities. Two subdivisions are located on opposite sides of Hay River, which is 100m wide. The company has to connect points P and Q with cable, where Q is on the north bank 1200m east of P. It costs \$40/m to lay cable underground and \$80/m to lay cable underwater. What is the least expensive way to lay the cable?



Two options

Option 1

Option 1

$80(100) + 40(1200)$

\$ 56000

Option 2

Diagonally to some point R then along shore

$RQ = \sqrt{(1200-x)^2 + 100^2}$

Cost of RQ

$\Rightarrow 80\sqrt{(1200-x)^2+100^2}$

Cost of $x = 40x$

Total Cost: $C(x) = 40x + 80\sqrt{(1200-x)^2+100^2}$ $\quad 0 \leq x \leq 1200$

$C'(x) = 40 + \dfrac{80}{2\sqrt{(1200-x)^2+100^2}} \cdot 2(1200-x)(-1)$

$C'(x) = 40 + \dfrac{40(2)(-1)(1200-x)}{\sqrt{(1200-x)^2+100^2}} = 40\left(1 + \dfrac{-2(1200-x)}{\sqrt{(1200-x)^2+100^2}}\right) = 0$

$\rightarrow 1 + \dfrac{-2(1200-x)}{\sqrt{(1200-x)^2+100^2}} = 0 \rightarrow 1 = \dfrac{2(1200-x)}{\sqrt{(1200-x)^2+100^2}} \rightarrow \sqrt{(1200-x)^2+100^2} = 2(1200-x)$

$\rightarrow (1200-x)^2 + 100^2 = 4(1200-x)^2 \rightarrow 100^2 = 3(1200-x)^2 \rightarrow 100 = \pm\sqrt{3}(1200-x)$

$1200 - x = \pm \dfrac{100}{\sqrt{3}}$

$x = 1200 \pm \dfrac{100}{\sqrt{3}} \doteq 1142 \text{ or } 1258$

outside domain

Homework Problems

- Section 4.4: #1 – 6, any 3 more questions

Crit Pts: $[0, 1200]$ and $x = 1142$

$C(0) = \$96333$

$C(1200) = \$56000$

$\boxed{C(1142) = 54928}$