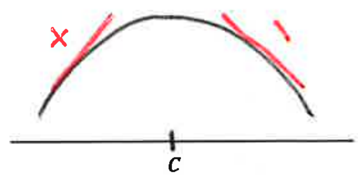
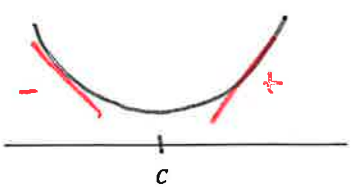
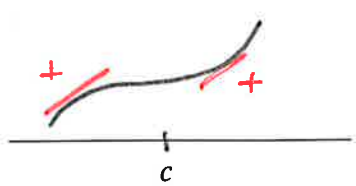
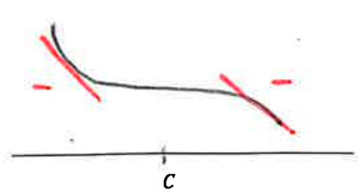
4.3 The First Derivative Test

We know from Section 4.1 that f is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. Therefore, we have the following test.

First Derivative Test

Let c be a critical number of a continuous function f.

1. If $f'(x)$ changes from **positive to negative** at c, then f has a **local maximum** at c.
2. If $f'(x)$ changes from **negative to positive** at c, then f has a **local minimum** at c.
3. If $f'(x)$ **does not change sign** at c, then f has **no maximum or minimum** at c.

Sign of $f'(x)$ to the left of c	**Sign of $f'(x)$ to the right of c**	**Graph**	**$f(c)$**
+	–		Local Maximum
–	+		Local Minimum
+	+		Neither
–	–		Neither

Ex 1: Find the local maximum and minimum values of $f(x) = x^3 - 3x + 1$.

$f'(x) = 3x^2 - 3$

$= 3(x^2 - 1)$

$= 3(x+1)(x-1)$

critical points: -1 and 1

$f(-1) = 3$ local max

$f(1) = -1$ local min

$f'(x)$ changes from positive to negative at -1. And negative to positive at 1

So...

$f'(x) > 0$

$x^2 - 1 > 0$

$x^2 > 1$

$|x| > 1$

$x < -1$ or $x > 1$

$f'(x) < 0$

$x^2 - 1 < 0$

$x^2 < 1$

$|x| < 1$

$-1 < x < 1$

Ex 2: Find the local maximum and minimum values of $g(x) = x^4 - 4x^3 - 8x^2 - 1$ and use the information to sketch the graph of g.

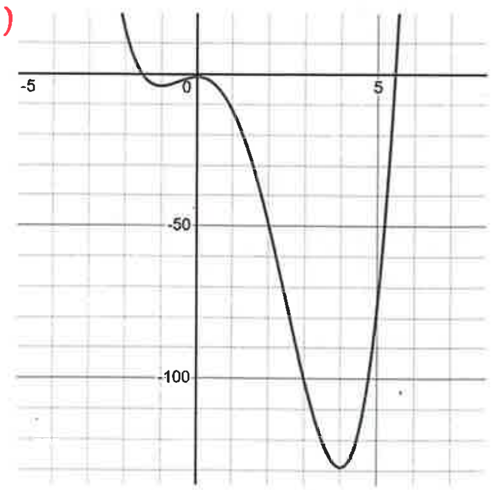
$g'(x) = 4x^3 - 12x^2 - 16x$

$= 4x(x^2 - 3x - 4)$

$= 4x(x-4)(x+1)$

Critical Numbers: $0, -1, 4$

Interval	$4x$	$x+1$	$x-4$	$g'(x)$	g
$x < -1$	$-$	$-$	$-$	$-$	dec $(-\infty, -1)$
$-1 < x < 0$	$-$	$+$	$-$	$+$	inc $(-1, 0)$
$0 < x < 4$	$+$	$+$	$-$	$-$	dec $(0, 4)$
$x > 4$	$+$	$+$	$+$	$+$	inc $(4, \infty)$



We see changes at $x = -1$ $x = 0$ $x = 4$

$g(-1) = -4$ local minimum

$g(0) = -1$ local maximum

$g(4) = -129$ absolute minimum

Ex 3: Find the critical numbers, intervals of increase and decrease, and local maximum and minimum values of the function:

$$f(x) = 2x - 3x^{\frac{2}{3}}$$

$$f'(x) = 2 - 2x^{-\frac{1}{3}} = 2 - \frac{2}{x^{\frac{1}{3}}} = \frac{2x^{\frac{1}{3}} - 2}{x^{\frac{1}{3}}} = \frac{2(x^{\frac{1}{3}} - 1)}{x^{\frac{1}{3}}}$$

critical values: $x = 0$
$x = 1$

Interval	$x^{\frac{1}{3}}$	$(x^{\frac{1}{3}} - 1)$	$f'(x)$	f
$x < 0$	$-$	$-$	$+$	inc $(-\infty, 0)$
$0 < x < 1$	$+$	$-$	$-$	dec $(0, 1)$
$x > 1$	$+$	$+$	$+$	inc $(1, \infty)$

$f(0) = 0$ local max (switches from inc to dec)

$f(1) = -1$ local min (switches from dec to inc)

In certain situations, the First Derivative Test can be used to find an **absolute** maximum or minimum.

First Derivative Test for Absolutely Extreme Values

Let c be a critical number of a continuous function f defined on an interval.

1. If $f'(x)$ is positive for all $x < c$ and $f'(x)$ is negative for all $x > c$, then $f(c)$ is the absolute maximum value.
2. If $f'(x)$ is negative for all $x < c$ and $f'(x)$ is positive for all $x > c$, then $f(c)$ is the absolute minimum value.

Ex 4: Find the absolute minimum value of the function

$$f(x) = x + \frac{1}{x}, \quad x > 0.$$

$f(x) = x + x^{-1}$

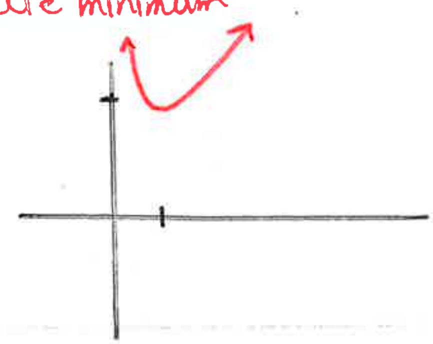
$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$

$f'(x) = 0$ when $x^2 - 1 = 0$ so $x = \pm 1$ but only 1 because

when $f'(x) > 0$ when $x^2 > 1$ or when $x > 1$

$f'(x) < 0$ when $0 < x < 1$ critical point: 1

So by the first derivative test, the absolute minimum value of f is $f(1) = 2$.



Homework Problems

- Section 4.3: #1ac, 2ade, 4bc