Section 4.3 – Solving Quadratics by Factoring and the Square Root Method

This Booklet Belongs to: Block:

Definition of a Quadratic Equation

An equation that can be written in the form:

$$\boldsymbol{ax^2 + bx + c = 0}$$

Where $a, b, and\ c$ are Real Numbers with $a \neq 0$

- The solutions of the quadratic equation $f(x) = ax^2 + bx + c = 0$ are called **zeros**, **roots, or solutions**
- The points where the graph **crosses the $\boldsymbol{x - axis}$** are **real** solutions, because $\boldsymbol{f(x)}$ **is $\boldsymbol{0}$**
- The $\boldsymbol{x - intercepts}$ are called the **real roots** of the quadratic function
- A quadratic function can cross the $x - axis$ either $\boldsymbol{0, 1, or\ 2\ times}$

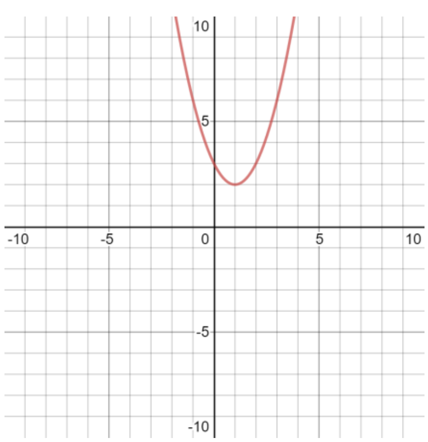
Real Zeros of a Quadratic Function

If $f(x)$ is a quadratic function and c is a real zero of $f(x)$, then the following statements are equivalent

1. $x = c\ is\ a\ \boldsymbol{zero}\ of\ the\ function\ f(x)$
2. $x = c\ is\ a\ \boldsymbol{root}\ of\ the\ function\ f(x)$
3. $x = c\ is\ a\ \boldsymbol{solution}\ of\ the\ equation\ f(x) = 0$
4. $(x - c)\ is\ a\ factor\ of\ the\ quadratic\ equation\ f(x)$
5. $(c, 0) is\ an\ x - intercept\ of\ teh\ graph\ of\ f(x)$

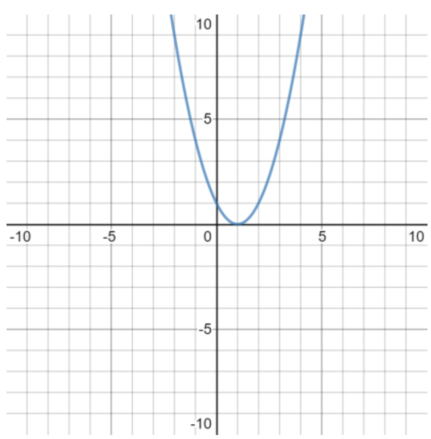
Example 1:

$f(x) = x^2 - 2x + 3$



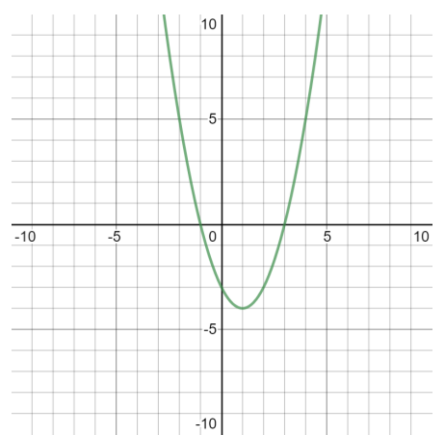
No zeros
(No real roots)

$f(x) = x^2 - 2x + 1$



One zero
(Double root/two roots equal)

$f(x) = x^2 - 2x - 3$



Two zeros
(Two unequal real roots)

Factoring Quadratics in the Form $x^2 + bx + c$

Consider this: $(x + a)(x + b) = x^2 + bx + ax + ab$

$$x^2 + (b + a)x + ab$$

- By looking at this we see that:
  - The first term is the product of x and x
  - The **coefficient of the middle term** is the **sum** of a and b
  - The last term is the **product** of a and b
- This leads us to the **general rule**:

When factoring $x^2 + bx + c$, look for **two factors of c**, that **multiply** to the **coefficient of the last term**, and **add** to the **coefficient of the middle term**.

Example: Factor $x^2 + 7x + 12$

Solution: What two numbers **add to 7** and **multiply to 12**?

- Integers that multiply to 12: $(1, 12)$ $(2, 6)$ $(3, 4)$ $(-1, -12)$ $(-2, -6)$ $(-3, -4)$
- Only integers 3 *and* 4 *add to* 7
- Therefore $x^2 + 7x + 12 = (x + 3)(x + 4)$
- We can check our answer using FOIL: $(x + 3)(x + 4)$

$$= x^2 + 3x + 4x + 12$$
$$= \mathbf{x^2 + 7x + 12}$$

Example: Factor $x^2 + 8 - 6x$

Solution: First arrange the polynomial in descending order of powers

- $x^2 + 8 - 6x = x^2 - 6x + 8$
- -4 and -2 add to -6 and multiply to $+8$
- Therefore: $x^2 - 6x + 8 = (x - 4)(x - 2)$
- We can check using FOIL

Example: Factor $5x^2 + 35x + 60$

Solution: Always look for a common factor first. The largest common factor is 5

Therefore: $5x^2 + 35 + 60 = 5(x^2 + 7 + 12)$

$$= 5(x + 3)(x + 4)$$

Example: Factor $-x^2 + 5x + 6$

Solution: First factor out -1, so that the coefficient of x^2 becomes $+1$.

- So $-x^2 + 5x + 6$ becomes $-(x^2 - 5x - 6)$, now factor $(x^2 - 5x - 6)$
- -6 and 1 multiply to -6 and add to -5
- Therefore $-x^2 + 5x + 6 = -(x^2 - 5x - 6) \quad = -(x-6)(x+1)$
- **Note the factors are: $(x - 6)(x + 1)$ and -1**

SUMMARY OF FACTORING POLYNOMIALS

1. Arrange the polynomial in descending order of powers

2. When the **last term** is **positive**, the **factors of c** are **both positive**, **or both negative**. If the **middle term** is **positive**, **both integers** are **positive**. If the **middle term** is **negative**, **both integers** are **negative**.

   Example: $x^2 + 7x + 12 = (x + 4)(x + 3)$

   - The last term is positive, and the middle term is positive, therefore the factors of 12 are both positive.
   - Opposite if the middle term was negative and the last positive.

3. When the last term is negative, the factors of c have opposite signs. The larger numeric value takes the sign of the coefficient of the middle term.

   Example: $x^2 - x - 6 = (x - 3)(x + 2)$

   - The last term is negative, therefore the signs of the factor of 6 are opposite of each other, and since the middle term is negative the larger numeric value has a negative sign.

   Example: $x^2 + 2x - 15 = (x + 5)(x - 3)$

   - The last term is negative, therefore the signs of the factor of 15 are opposite of each other, and since the middle term is positive the larger numeric value has a positive sign.

Factoring Quadratics in the Form $ax^2 + bx + c$

- There are a number of ways to factor equations in this form
- It is my humble opinion that the one shown below works the most efficiently
- In the last section we factored $ax^2 + bx + c$ where $\boldsymbol{a} = \mathbf{1}$. In this section $\boldsymbol{a}$ will have an integer value greater than 1.

The "AC" Method of Factoring

- The "AC" method is a technique used to factor trinomials
- A trinomial consists of three terms $(ax^2 + bx + c)$.

$a \quad b \quad c$

1. $6x^2 + 7x + 2$

1. Multiply the **coefficient of the a term** with the **coefficient of the c term** and rewrite the polynomial with the starting term now x^2

So $6x^2 + 7x + 2$ becomes $x^2 + 7x + 12$

3.

a) Find two numbers **that multiply** to $+12$ and **add** to $+7$

These are: $+3 \; and +4$

b) Rewrite the factored form of the new version of the Polynomial

$$(x+3)(x+4)$$

c) Now **divide the two factors** by the **original a term** and **simplify the fractions**

$$(x+\tfrac{3}{6})(x+\tfrac{4}{6}) \quad \rightarrow \quad (x+\tfrac{1}{2})(x+\tfrac{2}{3})$$

d) If you are not left with a denominator then you are done. If a denominator remains, rewrite it in front of the x term.

Can't simplify so write the 2 in front of the x

$$(x+\frac{1}{2})(x+\frac{2}{3})$$

Can't simplify so write the 3 in front of the x

4. Rewrite as the factored from: $(2x+1)(3x+2)$

This is the Factored Form.

Special Factors and Zero Product

- Factoring **Perfect Square Trinomials**

$0 = a^2 + 2ab + b^2 = (a + b)^2$ **Example:** $x^2 + 8x + 16 = (x + 4)^2$

$0 = a^2 - 2ab + b^2 = (a - b)^2$ **Example:** $x^2 - 8x + 16 = (x - 4)^2$

- Factoring a **Difference of Squares**

$0 = a^2 - b^2 = (a + b)(a - b)$ **Example:** $x^2 - 4 = (x - 2)(x + 2) = 0$

Example: $16x^2 - 25y^2 = (4x + 5y)(4x - 5y) = 0$

- The **Zero Product**

$0 = a^2 + ab = a(a + b)$ **Example:** $2x^2 + 6x = 0 \rightarrow 2x(x + 3) = 0$

So: $x = 0 \text{ or } x = -3$

Solving Using Factoring

Example: Solve the equation $x^2 - x - 6 = 0$

Solution:

$$(x - 3)(x + 2) = 0$$

$$(x - 3) = 0 \quad or \quad (x + 2) = 0$$

$$x = 3 \quad or \quad x = -2$$

Example: Solve the equation $3x^2 + 9x = 0$

Solution:

$$3x^2 + 9x = 0 \quad \rightarrow \quad 3x(x + 3) = 0$$

$$3x = 0 \quad or \quad (x + 3) = 0$$

$$x = 0 \quad or \quad x = -3$$

Example: Solve the equation $6x^2 - 7x - 5 = 0$

Solution:

$$6x^2 - 7x - 5 = 0 \quad \rightarrow \quad x^2 - 7x - 30 = 0$$

$$(x - 10)(x + 3) = 0 \quad \rightarrow \quad \left(x - \frac{10}{6}\right)\left(x + \frac{3}{6}\right) = 0$$

$$\left(x - \frac{5}{3}\right)\left(x + \frac{1}{2}\right) = 0 \quad \rightarrow \quad x = \frac{5}{3} \quad or \quad x = -\frac{1}{2}$$

Example: Solve the equation $x(3x+1)=2$

Solution:

$$x(3x+1)=2 \quad \rightarrow \quad 3x^2+x=2$$

$$3x^2+x-2=0 \quad \rightarrow \quad x^2+x-6=0 \quad \rightarrow \quad (x+3)(x-2)$$

$$\left(x+\frac{3}{3}\right)\left(x-\frac{2}{3}\right)=0 \quad \rightarrow \quad x=-\frac{3}{3}=-1 \quad or \quad x=\frac{2}{3}$$

Example: $\frac{x}{x-5}-\frac{3}{x+1}=\frac{30}{x^2-4x-5}$

Solution:

$$\frac{x}{x-5}-\frac{3}{x+1}=\frac{30}{x^2-4x-5}$$

- Factor the denominators

$$\frac{x}{x-5}-\frac{3}{x+1}=\frac{30}{(x-5)(x+1)}$$

- Identify LCM
- In this case: $(x-5)(x+1)$

$$(x-5)(x+1)\left[\frac{x}{x-5}-\frac{3}{x+1}=\frac{30}{(x-5)(x+1)}\right]$$

- Multiply each term by the LCM

$$(x+1)x-3(x-5)=30$$

- Reduce

$$x^2+x-3x+15-30=0$$

- Simplify

$$x^2-2x-15=0 \quad \rightarrow \quad (x-5)(x+3)$$

- Factor

$$\boldsymbol{x=5 \quad or \quad x=-3}$$

- Solve

- When finished **inset your answers** into the **original equation** to make sure they are valid
- The **denominator cannot equal zero** when you plug it in
- So it this case **reject $x=5$**

<u>Solving Quadratics Using the Square Root Method</u>

- The factor method is definitely the most efficient method of solving quadratics
- But not all quadratics can be factored easily of at all
- What we get are two methods depending on the situation
    - **The Square Root Method**
    - **The Quadratic Equation**
- The square root method in mainly used when $b = 0$ in the equation $ax^2 + bx + c = 0$
- Isolate the x^2 on the left side and square root the other side

Example:

$x^2 - 16 = 0$		$x^2 - 16 = 0$
$(x+4)(x-4)$	or	$x^2 = 16$
$x - 4 = 0 \; or \; x + 4 = 0$		$\sqrt{x^2} = \pm\sqrt{16}$
$x = 4 \; or \; x = -4$		$x = \pm 4$

The procedure on the right is the **SQUARE ROOT METHOD**

Solving a Quadratic Equation of the form $ax^2 + c = 0$

Step 1: Isolate the x^2 on the left side of the equation and the constant on the right
Step 2: Take the **square root of both sides**, the square root of the **constant has to be** $\pm$
Step 3: Simplify if possible
Step 4: Check the solution in the original equation

The **Square Root Property** is defined as follows:

The Square Root Property

The equation $x^2 = n$ has exactly 2 real solutions

$$x = \sqrt{n} \; and \; x = -\sqrt{n} \quad if \; n > 0. \; The \; solution \; are \; written \; x = \pm n$$

$$x = 0 \; if \; n = 0$$

$$x = \emptyset \; if \; n < 0 \; \; (we \; can't \; take \; the \; square \; root \; of \; a \; negative)$$

Example: Solve $4x^2 - 9 = 0$

Solution:

$4x^2 = 9$ - Add nine to both sides

$x^2 = \frac{9}{4}$ - Divide both sides by 4

$x = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2}$ - Square Root both sides (remember $\pm$)

Solutions are: $\boldsymbol{\frac{3}{2}\ and\ -\frac{3}{2}}$

Example: Solve $2x^2 + 7 = 0$

Solution:

$2x^2 = -7$ - Subtract seven from both sides

$x^2 = -\frac{7}{2}$ - Divide both sides by 2

$x = \pm\sqrt{-\frac{7}{2}} = \emptyset$ - $\emptyset$ denotes the '$\boldsymbol{empty\ set}$'

- We **can't square root a negative**

Solutions are: $\boldsymbol{None}$

Example: Solve $4x^2 - 7 = 0$

Solution:

$4x^2 = 7$ - Add seven to both sides

$x^2 = \frac{7}{4}$ - Divide both sides by 4

$x = \pm\sqrt{\frac{7}{4}} = \pm\frac{\sqrt{7}}{2}$ - Square Root both sides, and simplify

Solutions are: $\boldsymbol{\frac{\sqrt{7}}{2}\ and\ -\frac{\sqrt{7}}{2}}$

Example: Solve $(x+2)^2 = 10$

Solution:

$(x+2) = \pm\sqrt{10}$ - Square Root both sides first, the 'squared term' is isolated to start

$x = -2 \pm \sqrt{10}$ - Subtract both sides by 2

Solutions are: $\boldsymbol{-2+\sqrt{10}\quad and\quad -2-\sqrt{10}}$

Example: Solve $9(x-1)^2 = 13$

Solution:

$(x-1)^2 = \frac{13}{9}$	- Divide both sides by 9 to isolate the 'squared term'
$x - 1 = \pm\sqrt{\frac{13}{9}}$	- Square Root both sides
$x = 1 \pm \sqrt{\frac{13}{9}}$	- Add 1 to both sides
$x = 1 \pm \frac{\sqrt{13}}{3}$	- Simplify **Solutions are:** $\mathbf{1+\frac{\sqrt{13}}{3}}$ ***and*** $\mathbf{1-\frac{\sqrt{13}}{3}}$

Example: Solve $4(x+3)^2 + 11 = 0$

Solution:

$(x+3)^2 = -\frac{11}{4}$	- Subtract 11 and Divide both sides by 4 to isolate the 'squared term'
$x + 3 = \pm\sqrt{-\frac{11}{4}}$	- Square Root both sides
$x = \emptyset$	- **Can't Square Root a Negative**

Solutions are: ***None***

Section 4.3 – Practice Problems

Find the $x - intercepts$ by factoring the following equations.

1. $y = x^2 - 3x - 4$

2. $y = x^2 + x - 6$

3. $y = -x^2 + 4$

4. $y = -\frac{1}{2}x^2 - x + 4$

5. $y = 2x^2 + 5x - 3$

6. $y = -\frac{1}{3}x^2 + 3$

Solve the following equations. Check your solutions to make sure they are correct

7. $2x(4x - 3) = 0$

8. $(0.25y - 2)(0.2y + 1) = 0$

9. $x^2 = -x$

10. $x^2 + 1 = 0$

11. $4y^2 = y$

12. $3x^2 - x = 0$

13. $x^2 + 5x + 6 = 0$

14. $x^2 - 4x + 3 = 0$

15. $y^2 + y - 12 = 0$

16. $z^3 - 16z = 0$

17. $z(z-5)=-4$

18. $(x-12)(x+1)=-40$

19. $(y-6)(y+1)=-10$

20. $z^3-z^2=6z$

21. $x^3-3x=2x^2$

22. $\frac{x}{18}^2+\frac{x}{6}=1$

23. $(2x-1)^2 = 16$

24. $(3x+8)(x-1) = (x-1)(x+3)$

25. $(2y)^2 + (y+5)^2 = (2y+4)^2$

26. $6x^2(3x-1) - x(3x-1) = 2(3x-1)$

27. $\frac{1}{x}+\frac{3}{x-2}=\frac{5}{8}$

28. $\frac{4}{5}+y=\frac{4y-50}{5y-25}$

29. $\frac{4}{x^2-4}-\frac{1}{x-2}=1$

30. $\frac{1}{x-3}-\frac{12}{x^2-9}=1$

Complete the square and solve using the square root method, give exact answers

31. $x^2 + 6x = -5$

32. $y^2 - 5y + 3 = 0$

33. $z^2 - 8z + 3 = 0$

34. $x^2 - 2x - 1 = 0$

35. $y^2 + 4y + 2 = 0$

36. $z^2 + 2z + 7 = 0$

37. $5x^2 - 3x = 1$

38. $3x^2 - x = 3$

39. $3x^2 = -8x - 2$

40. $3x^2 = 6x + 2$

Section 4.3 – Answer Key

1. $(4,0)$ *and* $(-1,0)$	21. $x=0, x=3,$ *and* $x=-1$
2. $(-3,0)$ *and* $(2,0)$	22. $x=-6$ *and* $x=3$
3. $(-2,0)$ *and* $(2,0)$	23. $x=\frac{5}{2}$ *and* $x=-\frac{3}{2}$
4. $(-4,0)$ *and* $(2,0)$	24. $x=-\frac{5}{2}$ *and* $x=1$
5. $(-3,0)$ *and* $(\frac{1}{2},0)$	25. $y=3$
6. $(-3,0)$ *and* $(3,0)$	26. $x=\frac{2}{3}$ *and* $x=-\frac{1}{2}$ *and* $x=\frac{1}{3}$
7. $x=0$ *and* $x=\frac{3}{4}$	27. $x=8$ *and* $x=\frac{2}{5}$
8. $y=8$ *and* $y=-5$	28. $y=2$ *and* $y=3$
9. $x=0$ *and* $x=-1$	29. $x=-3$
10. *No Solution*	30. $x=0$ *and* $x=1$
11. $y=0$ *and* $y=\frac{1}{4}$	31. $x=-1$ *and* $x=-5$
12. $x=0$ *and* $x=\frac{1}{3}$	32. $y=\frac{5+\sqrt{13}}{2}$ and $y=\frac{5-\sqrt{13}}{2}$
13. $x=-3$ *and* $x=-2$	33. $z=4+\sqrt{13}$ and $z=4-\sqrt{13}$
14. $x=1$ *and* $x=3$	34. $x=1+\sqrt{2}$ and $x=1-\sqrt{2}$
15. $y=-4$ *and* $y=3$	35. $y=-2+\sqrt{2}$ and $y=-2-\sqrt{2}$
16. $z=0$ *and* $z=4$ *and* $z=-4$	36. *No Solution*
17. $z=4$ *and* $z=1$	37. $x=\frac{3+\sqrt{29}}{10}$ and $x=\frac{3-\sqrt{29}}{10}$
18. $x=4$ *and* $x=7$	38. $x=\frac{1+\sqrt{37}}{6}$ and $x=\frac{1-\sqrt{37}}{6}$
19. $y=1$ *and* $y=4$	39. $x=\frac{-4+\sqrt{10}}{3}$ and $x=\frac{-4-\sqrt{10}}{3}$
20. $z=0, z=3,$ *and* $z=-2$	40. $x=1+\sqrt{\frac{5}{3}}$ and $x=1-\sqrt{\frac{5}{3}}$