Section 4.3 – Rational Functions

- We solved Rationals in PC 11. We spent time identifying Domain Restrictions, but now we will see what that actually translates to when we try to graph these things.

> A **Rational Function** is a function of the form:
>
> $$f(x) = \frac{g(x)}{h(x)}$$
>
> $g(x)$ and $h(x)$ are both **polynomials where $h(x) \neq 0$.**

Examples: The **difference** between a **Function and Expression** is the **Equals Sign**

a)	$\frac{(x-4)}{(x+2)}$	This is a **Rational Expression**	Numerator/Denominator are Polynomials
b)	$y = \frac{(x-4)}{(x+2)}$	This is a **Rational Function**	Numerator/Denominator are Polynomials
c)	$y = 4x^3 + 2x^2 - 4x + 5$	This is a **Rational Function**	Numerator/Denominator are Polynomials
d)	$y = \frac{5x-7}{\sqrt{x+5}}$	This is **NOT** a Rational Function	Denominator is 1; still both Polynomials

What about the Domain Restrictions on the Denominator?

- Well $h(x)$ **cannot be 0**, any values that makes that happen give us either **Asymptotes or Holes**
- An **asymptote** is **a line** that your graph will get **infinitely close** to touching, but never will
- There are both **Vertical and Horizontal asymptotes**
- **Vertical Asymptotes** are literally **untouchable**
- **Horizontal Asymptotes** can be **crossed in certain situations** (you'll see this later)

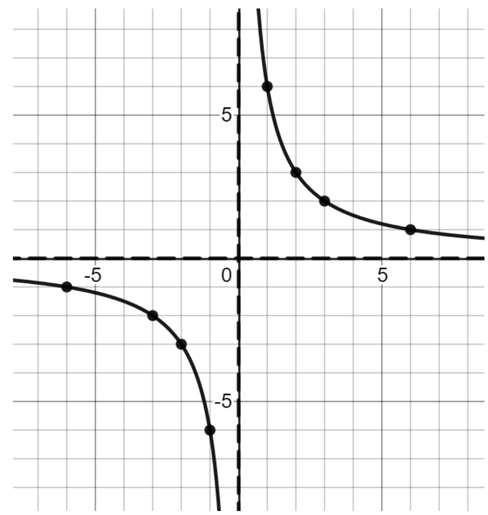
> **Definition of an Asymptote**
>
> An asymptote of a graph is a vertical or horizontal line that a part of the graph gets infinity close to, but never touches.

Example 1: Graph $f(x) = \dfrac{6}{x}$

Solution 1: We are not allowed to have a Denominator of 0, so $x \neq 0$, that is our Asymptote

- Since the asymptote is the vertical line $x = 0$ (the $y - axis$), the graph will approach but never touch
- There is a horizontal asymptote, the line $y = 0$, (the $x - axis$) we will see the behaviour and discuss these further later
- Create a table of values and plot points on either side of the vertical asymptote.

x	$f(x)$
0	Undefined
1	6
-1	-6
2	3
-2	-3
3	2
-3	-2
6	1
-6	-1



*Depending on the **complexity of the Denominator**, we may **need to factor** it to find all of the zeros*

Example 2: What are the Vertical Asymptotes of: $f(x) = \dfrac{x}{x^2 - 9}$

Solution 2: We are not allowed to have a Denominator of 0, so we need to factor our denominator

- $x^2 - 9$ is a difference of squares
- It factors to: $(x - 3)(x + 3)$

- So $x \neq 3$ and $x \neq -3$ There are Vertical Asymptotes at both of these points
- The lines $x = 3$ and $x = -3$ are displayed using dashes

Example 3: What are the Vertical Asymptotes of: $f(x) = \dfrac{2x + 3}{x^3 - 4x}$

Solution 3: We are not allowed to have a Denominator of 0, so we need to factor our denominator

- $x^3 - 4x$ is a difference of squares once we have factored out the x
- It factors to: $x(x^2 - 4) \quad \rightarrow \quad x(x - 2)(x + 2)$

- So $x \neq 0$ and $x \neq 2$ and $x \neq -2$, There are Vertical Asymptotes at all three of these points
- The lines $x = 0$ and $x = 2$ and $x = -2$ are displayed using dashes

Horizontal Asymptotes

- Horizontal asymptotes are also lines that cannot be touched (in most situations)
- But they have one very special difference….
- Horizontal Asymptotes occur as our graphs approach infinity in both directions on the $x - axis$

$$x \to \infty \ \ and \ \ x \to -\infty$$

- What this means is that horizontal asymptotes can be crossed, just not as we approach $\pm\infty$

We can look at the various scenarios of Rational Functions and Horizontal Asymptotes with respect to the degree of the Polynomials in the Numerator and Denominator.

<u>Horizontal Asymptotes of Rational Functions</u>

Consider the Rational Function

$$\boldsymbol{f(x) = \frac{g(x)}{h(x)}}$$

- If the **Degree of $\boldsymbol{g(x)} > \boldsymbol{h(x)}$**, then the **Horizontal Asymptote** is the **line $\boldsymbol{y = 0}$** $(\boldsymbol{x - axis})$
- If the **Degree of $\boldsymbol{g(x)} = \boldsymbol{h(x)}$**, then the **ratio of the leading coefficients** is the **Horizontal Asymptote**
- If the **Degree of $\boldsymbol{g(x)} < \boldsymbol{h(x)}$**, then there is **NO Horizontal Asymptote**

<u>Example 4:</u> Determine, if it exists, the horizontal asymptotes of the $f(x) = \dfrac{4x}{2x - 1}$

<u>Solution 4:</u> Rules are helpful, but if you can't remember them, then what. As you'll see in calculus **dividing each term** by the **variable** of the **highest power in the denominator** gives us the asymptote

$$\frac{4x}{2x - 1} \to \frac{\frac{4x}{x}}{\frac{2x}{x} - \frac{1}{x}}$$

And we are interested in the **behaviour** of the function as:
$\boldsymbol{x} \to \infty$

Cancel where we can and then **substitute** ∞ for any remaining $\boldsymbol{x - values}$

$$\frac{4}{2 - \frac{1}{x}} \to \frac{4}{2 - \frac{1}{\infty}}$$

$\frac{1}{\infty} = is\ a\ tiny\ small\ number$

So, we say: $\frac{1}{\infty} = 0$

$$\to \frac{4}{2 - \frac{1}{\infty}} \to \frac{4}{2 - 0} = \boldsymbol{2}$$

The **Horizontal Asymptote** is the **line $\boldsymbol{y = 2}$**, you'll notice it is the **ratio of the leading coefficients. Rule 2.**

<u>Example 5:</u> Determine, if it exists, the horizontal asymptotes of the $f(x) = \dfrac{3-4x}{x^2+2x-1}$

<u>Solution 5:</u> Divide each term by the variable of the **highest power in the denominator**

$$\frac{3-4x}{x^2+2x-1} \rightarrow \frac{\frac{3}{x^2}-\frac{4x}{x^2}}{\frac{x^2}{x^2}+\frac{2x}{x^2}-\frac{1}{x^2}} \rightarrow \frac{\frac{3}{x^2}-\frac{4}{x}}{1+\frac{2}{x}-\frac{1}{x^2}} \rightarrow \frac{\frac{3}{\infty^2}-\frac{4}{\infty}}{1+\frac{2}{\infty}-\frac{1}{\infty^2}}$$

$$\frac{\frac{3}{\infty^2}-\frac{4}{\infty}}{1+\frac{2}{\infty}-\frac{1}{\infty^2}} \rightarrow \frac{0-0}{1+0-0} = \frac{0}{1} = \mathbf{0}$$

The **Horizontal Asymptote** is the **line $\boldsymbol{y=0}$**, (the $x-axis$). **Rule 1.**

<u>Example 6:</u> Determine, if it exists, the horizontal asymptotes of the $f(x) = \dfrac{x^2+5x-6}{2x-1}$

<u>Solution 6:</u> Divide each term by the variable of the **highest power in the denominator**

$$\frac{x^2+5x-6}{2x-1} \rightarrow \frac{\frac{x^2}{x}+\frac{5x}{x}-\frac{6}{x}}{\frac{2x}{x}-\frac{1}{x}} \rightarrow \frac{x+5-\frac{6}{x}}{2-\frac{1}{x}} \rightarrow \frac{\infty+5-\frac{6}{\infty}}{2-\frac{1}{\infty}}$$

$$\frac{\infty+5-\frac{6}{\infty}}{2-\frac{1}{\infty}} \rightarrow \frac{\infty+5-0}{2-0} = \frac{\infty}{2} = \infty$$

The **Horizontal Asymptote DOES NOT EXIST**, **Rule 3.**

- Remembering the rules helps simplify this process, but understanding the behaviour as $x \rightarrow \infty$, has **significant application and connection to Limits in Calculus**.
- Also, we mentioned that **Horizontal Asymptotes can be crossed**. This is because we are only concerned with the **restriction of a Horizontal Asymptote as $\boldsymbol{x \rightarrow \pm\infty}$.**

We will see how graphs reflect these scenarios in Section 4.4

Finding the $\boldsymbol{x-intercept(s)\ and\ y-intercept}$

- We have been finding these intercepts since grade 9
- The $\boldsymbol{y-intercept}$ is where the function **crosses the $\boldsymbol{y-axis}$**, this happens **when $\boldsymbol{x=0}$**
- The $\boldsymbol{x-intercept(s)}$ is where the function **crosses the $\boldsymbol{x-axis}$**, this happens **when $\boldsymbol{y=0}$**
- Set the appropriate variable to 0 and solve for the desired information

__**Example 7:**__ Find the $x-intercept(s)$ and the $y-intercept$ of: $f(x)=\dfrac{x^2+2x-3}{x^2-9}$

__**Solution 7:**__ For the $x-intercept$ set $y=0$, for the $y-intercept$ set $x=0$

$\boldsymbol{y-intercept}$	$\boldsymbol{x-intercept}$
Let $x=0$, $$f(0)=\frac{(0)^2+2(0)-3}{(0)^2-9}$$ $$f(0)=\frac{-3}{-9}=\frac{1}{3}$$ $y-intercept\ is$: $\left(0,\frac{1}{3}\right)$	Let $f(x)=y=0$, $$y=\frac{x^2+2x-3}{x^2-9}$$ Only interested in when the Numerator is 0 $0=x^2+2x-3$ $0=(x+3)(x-1)\quad\rightarrow\quad x=-3,1$ $x-intercept\ is$: $(-3,0)\ and\ (1,0)$

__**Example 8:**__ Find the $x-intercept(s)$ and the $y-intercept$ of: $f(x)=\dfrac{x^4-16}{x^3+x}$

__**Solution 8:**__ For the $x-intercept$ set $y=0$, for the $y-intercept$ set $x=0$

$\boldsymbol{y-intercept}$	$\boldsymbol{x-intercept}$
Let $x=0$, $$f(0)=\frac{x^4-16}{x^3+x}$$ $$f(0)=\frac{0^4-16}{0^3-0}=\frac{-16}{0}=Undefined$$ $y-intercept\ is$: $\boldsymbol{None}$	Let $f(x)=y=0$, $0=\dfrac{x^4-9}{x^3+x}$ Only interested in when the Numerator is 0 $0=x^4-9$ $0=(x^2-3)(x^2+3)$ $0=\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)(x^2+3)\rightarrow x=\pm\sqrt{3}$ $x-intercept\ is$: $\left(-\sqrt{3},0\right)\ and\ (\sqrt{3},0)$

Example 9: Find the $x - intercept(s)$ and the $y - intercept$ of: $f(x) = \dfrac{x^2 - 4}{x^2 - x - 2}$

Solution 9: For the $x - intercept$ set $y = 0$, for the $y - intercept$ set $x = 0$

$\boldsymbol{y - intercept}$	$\boldsymbol{x - intercept}$
Let $x = 0$, $f(0) = \dfrac{(0)^2 - 4}{(0)^2 - (0) - 2}$ $f(0) = \dfrac{-4}{-2} = 2$ $y - intercept\ is{:}\ \ (0, 2)$	Let $f(x) = y = 0$, $y = \dfrac{x^2 - 4}{x^2 - x - 2}$ Only interested in when the Numerator is 0 $0 = x^2 - 4$ $0 = (x - 2)(x + 2) \quad \rightarrow \quad x = -2, 2$ $x - intercept\ is{:}\ \ (-2, 0)\ and\ (2, 0)$

Example 10: Find the $x - intercept(s)$ and the $y - intercept$ of: $f(x) = \dfrac{1}{x+1} - \dfrac{1}{x-1} + 2$

Solution 10: For the $x - intercept$ set $y = 0$, for the $y - intercept$ set $x = 0$, some algebra is required to get into a form that we can use efficiently; LCM in this case is $(x + 1)(x - 1)$

$$\frac{1}{x+1} - \frac{1}{x-1} + 2 \rightarrow \frac{(x-1) - (x+1) + 2(x-1)(x+1)}{(x+1)(x-1)} \rightarrow \frac{-2 + 2x^2 - 2}{(x+1)(x-1)} = \boldsymbol{\frac{2x^2 - 4}{(x+1)(x-1)}}$$

$\boldsymbol{y - intercept}$	$\boldsymbol{x - intercept}$
Let $x = 0$, $f(0) = \dfrac{2(0)^2 - 4}{(0+1)(0-1)}$ $f(0) = \dfrac{-4}{-1} = 4$ $y - intercept\ is{:}\ \ (0, 4)$	Let $f(x) = y = 0$, $y = \dfrac{2x^2 - 4}{(x+1)(x-1)}$ Only interested in when the Numerator is 0 $0 = 2x^2 - 4$ $2x^2 = 4 \rightarrow x^2 = 2 \quad x = -\sqrt{2}, \sqrt{2}$ $x - intercept\ is{:}\ \ (-\sqrt{2}, 0)\ and\ (\sqrt{2}, 0)$

Holes in Rational Functions

- As mentioned earlier, we can have **holes in our graph**
- This occurs when we **have restrictions in our denominator**, thus creating a vertical asymptote, but then **this asymptote values factors and cancels out**
- The **cancelled-out factor creates a hole** instead of an asymptote

Example 11: Graph the function $f(x) = \dfrac{x+2}{x^2 - x - 6}$

Solution 11: Consider the denominator: $x^2 - x - 6$, it factors to $(x-3)(x+2)$, this would give us two vertical asymptotes at $x = 3$ and $x = -2$

Look what happens when we see the entire function, with a factored denominator.

$$f(x) = \frac{x\ +\ 2}{x^2\ -\ x\ -\ 6} = \frac{x+2}{(x-3)(x+2)} \quad \rightarrow \quad \frac{\cancel{(x+2)}}{(x-3)\cancel{(x+2)}}$$

This reduces to:

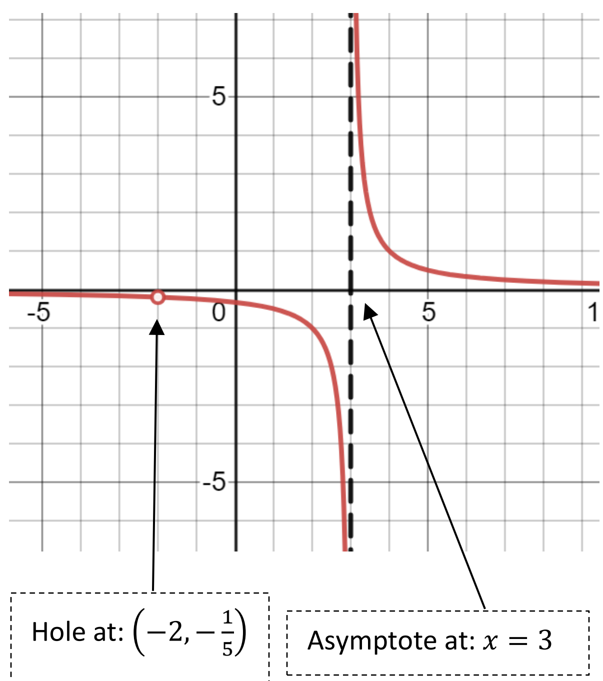
$$f(x) = \frac{1}{(x-3)}$$

- One of our denominator restrictions has cancelled out. This leaves us with only one asymptote at $x = 3$.
- $x = -2$ is still a Domain Restriction so it is still not allowed to exist on our graph, we still need to find the corresponding $y - value$ of the coordinate.

$$f(-2) = \frac{1}{(-2-3)} = -\frac{1}{5}$$

So, what this means is that we will have **a Hole at the coordinate**:

$$\left(-2, -\frac{1}{5}\right)$$



Example 12: Determine the vertical asymptotes and the holes of: $f(x)=\dfrac{x-3}{x^2-9}$

Solution 12: Factor the denominator to see if anything cancels

$$f(x)=\frac{x-3}{x^2-9}$$

$$f(x)=\frac{x-3}{(x-3)(x+3)}$$

$$f(x)=\frac{\cancel{(x-3)}}{\cancel{(x-3)}(x+3)}$$

$$f(x)=\frac{1}{(x+3)}$$

Vertical Asymptote at the remaining factor: $(x+3)$

Vertical Asymptote at: $\boldsymbol{x=-3}$

Hole at $x-coordinate$ of 3

Find the $y-coordinate$

$f(3)=\dfrac{1}{(3+3)}=\dfrac{1}{6}$ **Hole at:** $(\mathbf{3},\dfrac{\mathbf{1}}{\mathbf{6}})$

Example 13: Determine the vertical asymptotes and the holes of: $f(x)=\dfrac{x^2+8x+12}{x+6}$

Solution 13: Factor the denominator to see if anything cancels

$$f(x)=\frac{x^2+8x+12}{x+6}$$

$$f(x)=\frac{(x+2)(x+6)}{(x+6)}$$

$$f(x)=\frac{(x+2)\cancel{(x+6)}}{\cancel{(x+6)}}$$

$$f(x)=x+2$$

The denominator cancelled out we have no Vertical Asymptote

Vertical Asymptote at: $\boldsymbol{NONE}$

Hole at $x-coordinate$ of -6

Find the $y-coordinate$

$f(-6)=(-6+2)=-4$ **Hole at:** $(\mathbf{-6},\mathbf{-4})$

In the next Section we put all of this together and look at graphing these Rational Functions

Section 4.3– Practice Problems

1. Answer the following questions to lockdown your vocabulary.

a) A rational expression is defined as the ___________ of two polynomials with the ____________ not equal to zero.

b) A function defined by $f(x) = \frac{g(x)}{h(x)}$ with $g(x)$ and $h(x)$ __________ functions and $h(x) \neq 0$, is called a _________ function

c) To determine the excluded values of the domain of a rational function, we find the values for where the ________________ is equal to _______________

d) A vertical line that a graph approaches but never touches is called a ___________ __________

e) A horizontal line that a graph approached as $x \to \pm\infty$ is called a _______________ _______________

f) The graph of a Rational Function $f(x) = \frac{3}{x-2}$ will have a vertical asymptote of ________________ and a horizontal asymptote of _____________

2. Find the Domain, the $x - intercept(s), y - intercepts$, and any $holes$ for the following

a) $y = \frac{3x-9}{4x+12}$

b) $y = \frac{(x+6)(x+3)}{(x-2)^2}$

c) $y = \dfrac{x^2 - 8x - 9}{x^2 - x - 6}$

d) $y = \dfrac{(x^2-1)\,(x + 1)}{x^3}$

e) $y = \dfrac{x + 2}{x^2 + 4}$

f) $y = \dfrac{-3x^2 + 12}{x^2 - 9}$

g) $y = \dfrac{4}{(x+4)^2}$

h) $y = \dfrac{-x^2+9}{-2x^2+8}$

i) $y = \dfrac{2+x}{x^2+4}$

j) $y = \dfrac{x^2-3x-4}{4+3x-x^2}$

3. Find the $y - intercept$ and $x - intercept(s)$ vertical and horizontal asymptotes of the following

a) $y = -\frac{4}{x}$

b) $y = -\frac{3}{x^2}$

c) $y = 1 + \frac{1}{x}$

d) $y = 2 - \frac{1}{x}$

e) $y = -4 + \frac{1}{x^2}$

f) $y = -1 - \frac{1}{x^2}$

g) $y = -\frac{1}{x+1}$

h) $y = -\frac{2}{(x+1)^2}$

i) $y = 3 + \frac{2}{x-1}$

j) $y = 1 - \frac{2}{(x+1)^2}$

4. What are the Vertical Asymptote, Horizontal Asymptote, and any holes of the following.

a) $y = \frac{1}{x}$

b) $y = \frac{2}{x+3}$

c) $y = \dfrac{1}{x^2 - 7x + 12}$

d) $y = \dfrac{x^2}{x^2 - 9}$

e) $y = \dfrac{x}{x^2 + 1}$

f) $y = \dfrac{x^3}{x^2 - x - 20}$

g) $y = \frac{x^2 + 3x - 1}{4 - x^2}$

h) $y = \frac{2x^3 - 18x}{x^3 - 3x^2 - 4x}$

i) $y = \frac{x^2 - 4}{2x^3 + 7x^2 - 4x}$

j) $y = \frac{9 - 6x}{4x^2 - 9}$

k) $y = \frac{16x - x^3}{2x^3 + 7x^2 - 4x}$

l) $y = 1 - \frac{3}{x^2 - 1}$

See Website for Detailed Answer Key of the Remainder of the Questions

Extra Work Space