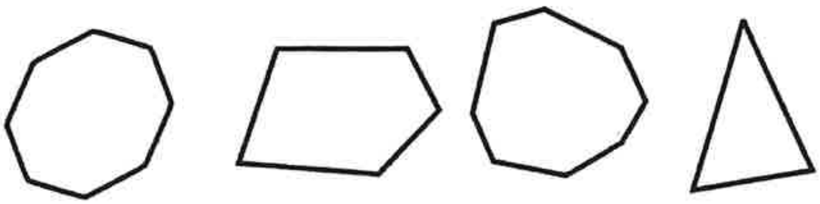
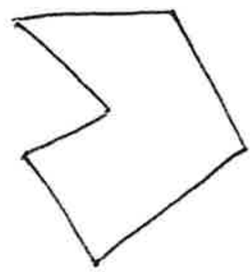
Section 4.3 – Polygons and Area-Volume Relationships

- A Polygon is a figure formed by connecting **line segments** at the **endpoints**.
- Each **line segment** is a **side**
- Each **endpoint** where the sides meet is called a **vertex**
- We will only discuss **Convex Polygons** in is section
- Convex polygons look like these...

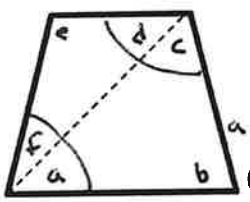
Concave Polygon





Interior Angle Sum of a Polygon

- As proven in the last section we know that triangles have **angles that add to 180°**
- If you **draw all the diagonals** from just **one vertex** of a polygon, you can figure out the sum of the interior angles.

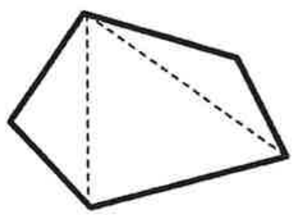


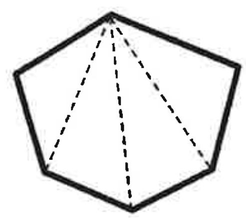
$a+b+c=180°$

$d+e+f=180°$

$a+b+c+d+e+f=360°$

$(a+f)+b+(c+d)+e=360°$





4 sides
2 triangles
Interior angle sum $= (2)180°$
$= 360°$

5 sides
3 triangles
Interior angle sum $= (3)180°$
$= 540°$

6 sides
4 triangles
Interior angle sum $= (4)180°$
$= 720°$

$\Sigma i = (2)\,180° = (4-2)\,180°$

$\Sigma i = (3)\,180° = (5-2)\,180°$

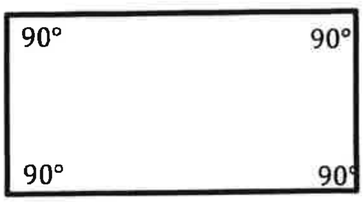
$\Sigma i = (4)\,180° = (6-2)\,180°$

Interior Angle Sum of an n-sided Polygon

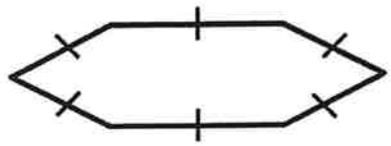
The sum of the interior angles of a polygon with n sides is:

$$\Sigma\, i = (n-2)180° \qquad (n > 2)$$

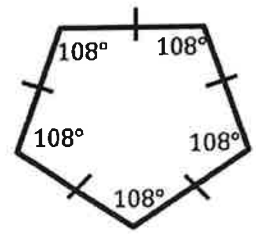
- A polygon can be equiangular (all angles the same) or
- A polygon can be equilateral (all sides the same)
- If it is both, it is called a **REGULAR POLYGON**



Equiangular Quadrilateral



Equilateral Hexagon



Regular Pentagon

For regular polygons, we can determine the measure of each interior angle.

$$\Sigma i = (n-2)180°$$

$$\frac{\Sigma i}{n} = \frac{(n-2)180°}{n}$$

$$i = \frac{(n-2)180°}{n}$$

$$i = \frac{(5-2)180°}{5} = 108°$$

Measure of Each Interior Angle of a Regular Polygon

- The measure of each interior angle of a **REGULAR** $n-sided$ polygon is:

$$i = \frac{(n-2)180°}{n} \qquad (n > 2)$$

Example: Find the measure of each interior angle of a 10-sided regular polygon (Decagon)

Solution:

$$i = \frac{(n-2)180°}{n}$$

$$i = \frac{(10-2)180°}{10}$$

$$i = \frac{(8)180°}{10}$$

$$= 144°$$

Example: Find the number of sides of a regular polygon that has interior angles of 165°

Solution:

$$i = \frac{(n-2)180°}{n}$$

$$165° = \frac{(n-2)180°}{n}$$

$$165°n = 180°n - 360°$$

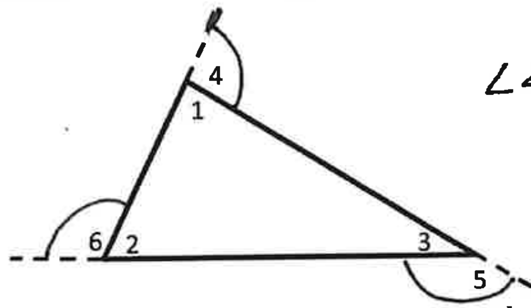
$$-15°n = -360°$$

$$n = \frac{-360°}{-15°}$$

$$= 24$$

Exterior Angles of a Polygon

- A Polygon has both interior and exterior angles



$\angle 4 + \angle 5 + \angle 6 = 360°$

$Angles\ 1, 2, and\ 3\ are\ interior\ angles$

$Angles\ 4, 5, and\ 6\ are\ exterior\ angles$

- At **each vertex** of the polygon the **interior and exterior angles add up** to **180° (supplementary angles)**
- If the Polygon has $\boldsymbol{n}$ ***sides***, then there are $\boldsymbol{n}$ **pairs of angles of 180°**
- The **sum** of these angles in **180°$\boldsymbol{n}$**
- So... $i + e = 180°$

$\Sigma(i + e) = 180° n$

$\Sigma i + \Sigma e = 180° n$

& since $\Sigma i = (n-2)180°$

$(n-2)180° + \Sigma e = 180° n$

$180° n - 360° + \Sigma e = 180° n$

$\Sigma e = 360°$

Exterior Angles of a Polygon

- The sum of exterior angles of a polygon is always 360°

$$\Sigma e = 360$$

- Each exterior angle of a **REGULAR** polygon is:

$$e = \frac{360°}{n}$$

Example: Find the interior angle of a 9-sided regular polygon

Solution:

Method 1 –

$e = \frac{360°}{n}$

$e = \frac{360°}{9}$

$e = 40°$

now, $i + e = 180°$

$i + 40° = 180°$

$i = 140°$

Method 2 –

$i = \frac{(n-2)180°}{n}$

$= \frac{(9-2)180°}{9}$

$i = \frac{(7)180°}{9}$

$i = 140°$

Example: What type of a regular polygon has an interior angle 3 times the exterior angle?

Solution:

$i + e = 180°$

$3e + e = 180°$

$4e = 180°$

$e = 45°$

Using the Exterior Angle formula:

$e = \frac{360°}{n}$

$45° = \frac{360°}{n}$

$n = \frac{360°}{45°}$

$n = 8$

The Polygon is an octagon

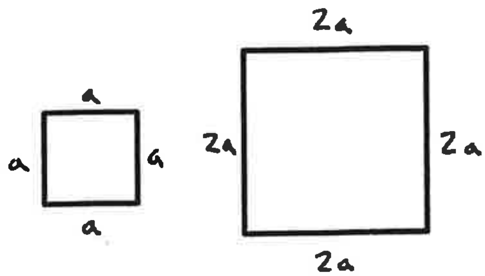
Classifying Polygons

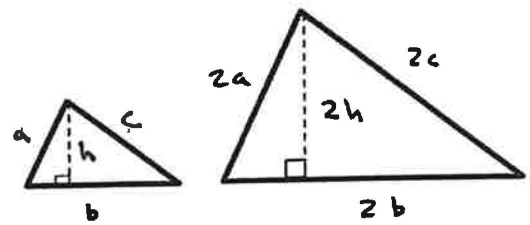
Sides	**Name**
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon

Sides	**Name**
9	Nonagon
10	Decagon
11	Undecagon
12	Dodecagon
n	$n - gon$

Practice Problems #1-5

Perimeter of Similar Figures





$F_s : F_L$

$a : 2a$

$1 : 2$

$P_s : P_L$

$4a : 8a$

$1 : 2$

$F_s : F_L$

$a : 2a$

$1 : 2$

$P_s : P_L$

$a+b+c : 2a+2b+2c$

$a+b+c : 2(a+b+c)$

$1 : 2$

If the scale factor of the two similar figures is $a: b$, then the ratio of their perimeters is $a: b$

$F_s : F_L$ $\quad$ $P_s : P_L$

$a : b$ $\quad$ $a : b$

Example: If two circles have radii 8 cm and 18 cm, what is the ratio of the circumferences? $C = 2\pi r$

$F_s : F_L$

$8 : 18$

$4 : 9$

$P_s : P_L$

$4 : 9$

$C_s = 2\pi(8)$ $\quad$ $C_L = 2\pi(18)$

$C_s : C_L$

$\cancel{2\pi}(8) : \cancel{2\pi}(18)$

$4 : 9$

Example: A circle has a circumference of 20 cm. It is scale down by a scale factor of 5. What is the circumference of the new circle? What is the new radius?

$F_s : F_L$

$1 : 5$

$a : b$

$P_s : P_L$

$P_s : 20\text{ cm}$

$a : b$

$$\frac{P_s}{P_L} = \frac{a}{b}$$

$$\frac{P_s}{20} = \frac{1}{5}$$

$$P_s = \frac{20(1)}{5}$$

$P_s = 4$ cm circumference

$C = 2\pi r$

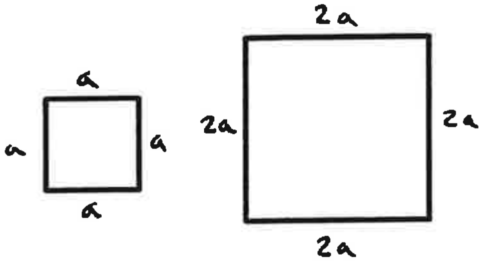
$4\text{ cm} = 2\pi r$

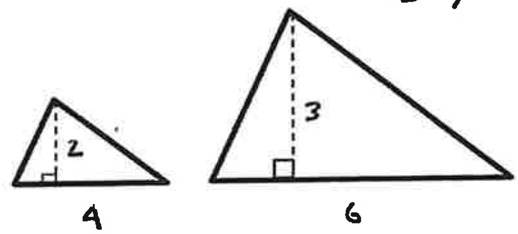
$$r = \frac{4\text{cm}}{2\pi} = 0.6366\text{ cm}$$

Area of Similar Figures

$A_s = \frac{4(2)}{2}$
$= 4$

$A_L = \frac{6(3)}{2}$
$= 9$





$F_s : F_L$
$a : 2a$
$1 : 2$

$A_s : A_L$
$a^2 : (2a)^2$
$a^2 : 4a^2$
$1 : 4 \Rightarrow 1^2 : 2^2$

$F_s : F_L$
$2 : 3$

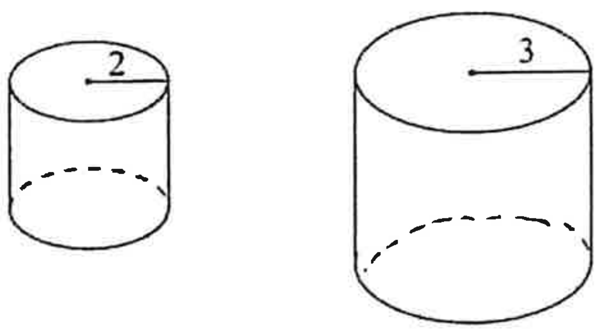
$A_s : A_L$
$4 : 9$
$2^2 : 3^2$

If the scale factor of the two similar figures is $a : b$, then the ratio of their areas is $a^2 : b^2$

$F_s : F_L$
$a : b$

$A_s : A_L$
$a^2 : b^2$

Example: The two cylinders are similar. If the surface area of the largest cylinder is $54\pi cm^2$, find the surface area of the smaller cylinder.



Solution:

$F_s : F_L$
$2 : 3$
$a : b$

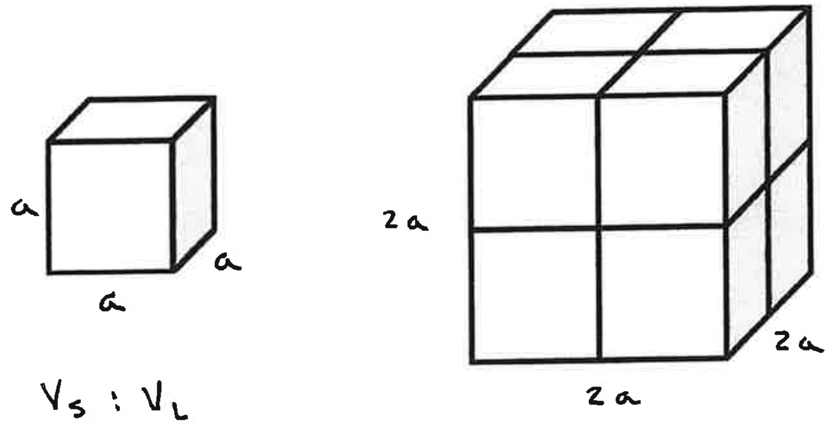
$A_s : A_L$
$a^2 : b^2$
$2^2 : 3^2$
$4 : 9$

$$\frac{A_s}{A_L} = \frac{a^2}{b^2}$$

$$\frac{A_s}{54\pi cm^2} = \frac{4}{9}$$

$$A_s = \frac{4(54\pi cm^2)}{9}$$
$$= 24\pi cm^2$$

Volume of Similar Figures



$F_S = F_L$ $V_S : V_L$

$a : 2a$ $a^3 : (2a)^3$

$1 : 2$ $a^3 : 8a^3$

$1 : 8 \Rightarrow 1^3 : 2^3$

If the scale factor of the two similar figures is $a: b$, then the ratio of their volume is $a^3: b^3$

$F_S : F_L$ $V_S : V_L$

$a : b$ $a^3 : b^3$

Example: Two spheres are made of the same material. The smaller sphere has a mass of 5 kg. the ratio of their radii is 2 to 7. What is the mass of the larger sphere? $m = \rho V$

Solution:

$F_S : F_L$ $V_S : V_L$

$2 : 7$ $a^3 : b^3$

$a : b$ $2^3 : 7^3$

$8 : 343$

$$\frac{V_S}{V_L} = \frac{a^3}{b^3} \quad \Big| \quad \frac{m_S}{m_L} = \frac{\cancel{\rho} V_S}{\cancel{\rho} V_L}$$

$$\frac{m_S}{m_L} = \frac{a^3}{b^3}$$

$$\frac{5\,kg}{m_L} = \frac{8}{343} \Rightarrow m_L = 214.375\ kg$$

Example: Two spheres have a surface area ratio of 25 cm² to 64 cm². What is the ratio of their volumes?

Solution:

$A_S : A_L$ $F_S : F_L$ $V_S : V_L$

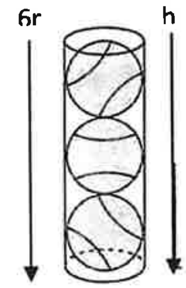
$25\,cm^2 : 64\,cm^2$ $a : b$ $a^3 : b^3$

$5^2 : 8^2$ $5 : 8$ $5^3 : 8^3$

$a^2 : b^2$ $125 : 512$

$125\,cm^3 : 512\,cm^3$

Example: A can of tennis balls usually holds 3 balls. What is the ratio of air in the can to volume of the can?



Solution:

Volume of a Cylinder: $V = \pi r^2 h$ Volume of Sphere: $V = \frac{4}{3}\pi r^3$

- 3 balls would have a height of 6 $radii$
- So the Volume of the can is:

$$V = \pi r^2 h$$
$$= \pi r^2 (6r)$$
$$= 6\pi r^3$$

- $Volume\ of\ air = Volume\ of\ the\ can\ -\ volume\ of\ the\ 3\ tennis\ balls$

$$= 6\pi r^3 - 3\left(\frac{4}{3}\pi r^3\right)$$
$$= 6\pi r^3 - 4\pi r^3$$
$$= 2\pi r^3$$

$$\frac{Volume\ of\ air}{Volume\ of\ can} = \frac{2\pi r^3}{6\pi r^3}$$
$$= \frac{1}{3}$$

$\rightarrow$ $The\ can\ is\ \frac{1}{3}\ air$

The Square-Cube Law

- From the ratios we've seen above, this explains that as the **length of a side increases** the **Surface Area increases** by the **square of its multiplier**, where the **volume increases by the cube of its multiplier**
- This demonstrates why if an animal was scaled up it's **weight would increase faster** than the cross section of its muscles, making it too heavy to support itself

Practice Problems #6 -10

Section 4.3 – Practice Problems

1. Find the sum of the interior angles of the polygons for each number of sides.

a) 20

b) 17

c) 39

d) 23

2. One interior angle of a regular polygon is given. Find the number of sides.

a) 156°

b) 144°

c) 160°

d) 179°

3. The number of sides of a regular polygon is given. Calculate the measure of each interior angle.

a) 4

b) 17

c) 8

d) 13

4. The sum of the interior angles of a regular polygon is given. Calculate the measure of each exterior angle.

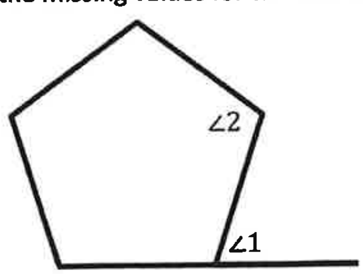
a) 2880°

b) 1620°

c) 3780°

d) 3420°

5. Find the missing values for the following polygons.



$\angle 1 =$

$\angle 2 =$

6. If two circles have radii $8\ and\ 12$, what is the ratio of the circumferences? Areas?

$$C = 2\pi r \quad and\ A = \pi r^2$$

7. If the area of two circles are 16π and 36π, what is the ratio of the radii? Of the circumferences?

8. Two similar cylinders have lateral (sides) surface areas of $36\pi\ and\ 64\pi$. Find the ratios of their heights. SA of Cylinder: $2\pi r^2 + 2\pi rh$; Lateral SA: $2\pi rh$

9. Two spheres of the same density have a surface area of $16\pi cm^2$ and $36\pi cm^2$. If the small ball weigh $6kg$, what does the larger sphere weigh? V of Sphere: $\frac{4}{3}\pi r^3$

10. Two identical cylinders are tightly packed side by side in a box $12cm\ by\ 6cm$ and $4cm\ in\ height.$ Find the ratio of the volume of the box to the volume of the cylinders.

Answer Key – Section 4.1

Please see Section 4.1 on the Website for Detailed Solutions

1. $Angle\ 1: 80°; Angle\ 2: 80°$	2. $Angle\ 1: 60°$
3. $Angle\ 1: 100°; Angle\ 2: 100°$	4. $Angle\ 1: 65°; Angle\ 2: 115°$
5. $Angle\ 1: 20°; Angle\ 2: 60°; Angle\ 3: 120°$	6. $Angle\ 1: 55°; Angle\ 2: 15°$
7. $Angle\ 1: 120°; Angle\ 2: 60°$	8. $Angle\ 1: 35°; Angle\ 2: 35°;\ Angle\ 3: 55°$
9. $Angle\ 1: 57°; Angle\ 2: 123°; Angle\ 3: 123°$	10. $Angle\ 1: 45°; Angle\ 2: 70°; Angle\ 3: 70°; Angle\ 4: 65°$
11. $Angle\ 1: 65°; Angle\ 2: 115°$	12. $Angle\ 1: 20°; Angle\ 2: 110°$
13. $Angle\ 1: 121°$	14. $Angle\ 1: 139°$
15. $Angle\ 1: 130°; Angle\ 2: 25°; Angle\ 3: 65°$	16. $Angle\ 5y: 100°; Angle\ 4y: 80°$

Answer Key – Section 4.2

See Website for All Solutions

Answer Key – Section 4.3

1. a) 3240° b) 2700° c) 6660° d) 3780°	2. a) 15 b) 10 c) 18 d) 360
3. a) 90° b) 158.8° c) 135° d) 152.3°	4. a) 20° b) 32.7° c) 15.7° d) 17.1°
5. $Angle\ 1: 72°; Angle\ 2: 108°$	6. 2 : 3; 4 : 9
7. 2 : 3; 2 : 3	8. 3 : 4
9. $20.25kg$	10. $4: \pi$