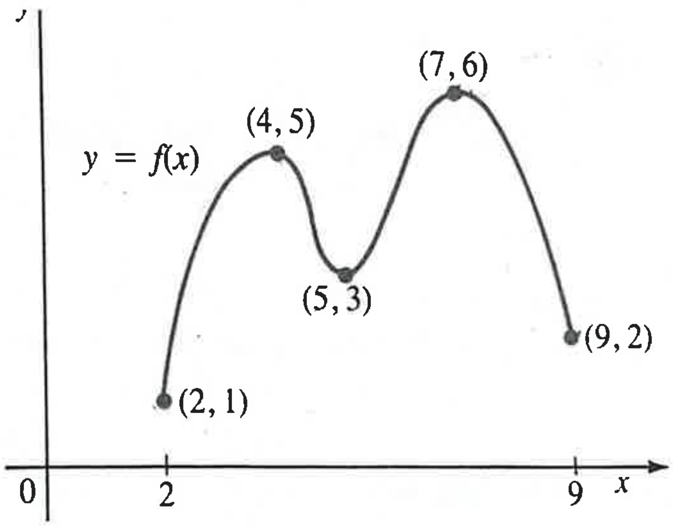
4.2 Maximum and Minimum Values

Below you see the graph of f. Notice that it has a highest point at $(7, 6)$, so the largest value taken from the function is $f(7) = 6$. We say that this point is the **absolute maximum**. The lowest point on the function occurs at $(2, 1)$, so the smallest value taken from the function is $f(2) = 1$. We say that this point is the **absolute minimum.**

Note: We will see different notation for maximum and minimum functions when the Domain is not restricted.

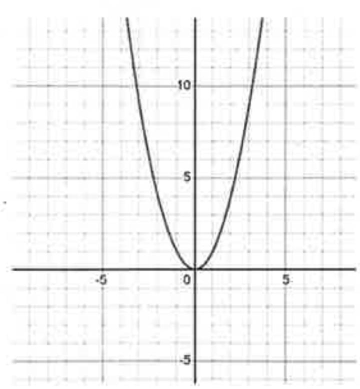


In general, a function has an **absolute maximum** at point c if $f(c) \geq f(x)$ for all x in the Domain of f, and the number $f(c)$ is called the **maximum value** of f. Likewise, a function has an **absolute minimum** at point c if $f(c) \leq f(x)$ for all x in the Domain of f, and the number $f(c)$ is called the **minimum value** of f.

The **extreme values** of f are the maximum and minimum values.

We will examine some graph below and discuss the difference between **absolute** and **local minimum** and **maximums**

Ex 1: If $f(x) = x^2$, then $f(x) \geq 0$ for all x since $x^2 \geq 0$. What are the absolute and local max/min values?



We see that $f(0) = 0$

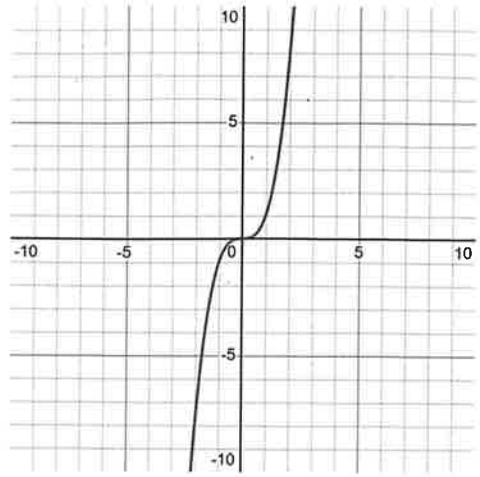
and $f(x) \geq f(0)$ for all x

so $f(0) = 0$ is the absolute and local minimum

- This function is continuous with no Domain Restriction or end points so there is no maximum value.

Ex 2: If $f(x) = x^3$, what are the absolute and local max/min values?

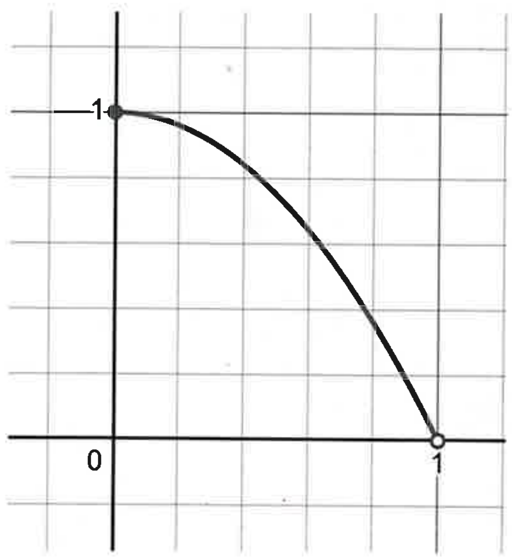
For this function there is no absolute or local maximum or minimum.



Ex 3: Consider the graph of the function $f(x) = 1 - x^2, 0 \leq x < 1$. What are the maximum/minimum values (local and absolute) if any?

We can see that $f(0) = 1$ is the maximum value

However, we have no minimum value. It takes on values very close to 0 but it never gets to 0.



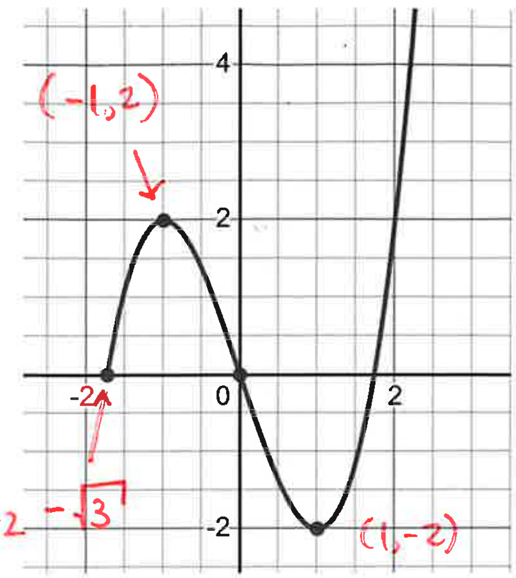
Ex 4: Consider the graph of the function $f(x) = x^3 - 3x, x \geq -\sqrt{3}$. What are the maximum/minimum values (local and absolute) if any?

From this graph:

$f(-1) = 2$

$f(-\sqrt{3}) = 0$

$f(1) = -2$



We can see that we have

a local maximum at $f(-1) = 2$

a local/absolute minimum at $f(1) = -2$

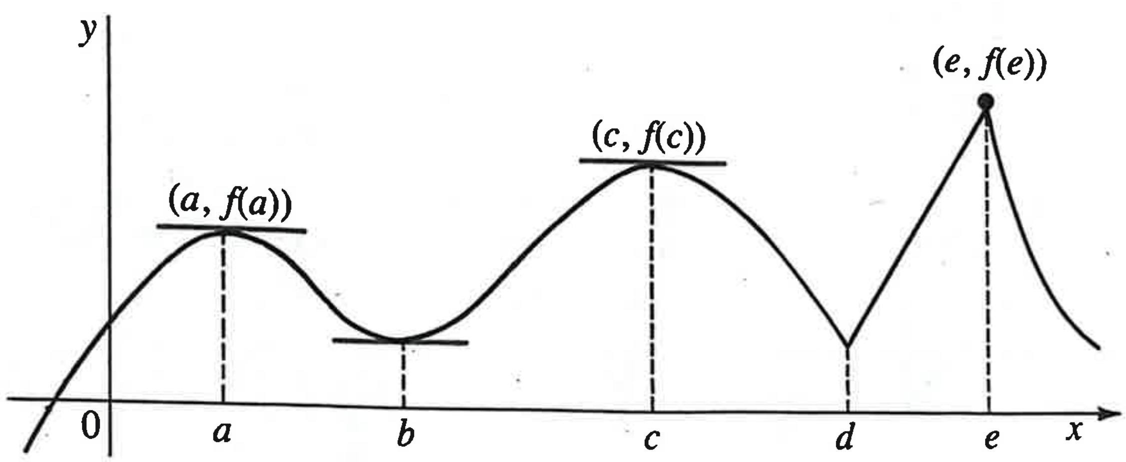
but no absolute maximum.

We see from the examples previous that function do not necessarily need to have maximum and minimum values. On a **continuous function over a closed interval** $[a, b]$ however, it can be proved that **there are both an absolute maximum and absolute minimum.**

Looking at the figure below, we can see that if we indeed have maximums and minimums, they will be found when the tangent lines are horizontal to our function f. In other words, when $f'(c)$ at a given c is equal to 0. This was first proposed and generalized by French Mathematician Pierre Fermat $(1601 - 1665)$.

Fermat's Theorem

If f has a local maximum or minimum at c, then either $f'(c) = 0$ or $f'(c)$ does not exist.



We do need to be cautious as **Fermat's Theorem does not automatically produce** local maximums of minimums (see the Example 2, the Cubic Function). But what Fermat's Theorem does say is that is we are looking for these extreme values, we should **start by looking** at these values for c **where $f'(c) = 0$ or $f'(c)$ does not exist**. These are known as **Critical Numbers.**

> A **critical number** of a function is a number c in the Domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Ex 5: Find the critical numbers of the following functions.

a) $f(x) = x^3 + 6x^2 + 9x + 2$

b) $f(x) = |x|$

a) $f'(x) = 3x^2 + 12x + 9$

$= 3(x^2 + 4x + 3)$

$= 3(x+1)(x+3)$

$0 = 3(x+1)(x+3)$

at $x = -1$

$x = -3$

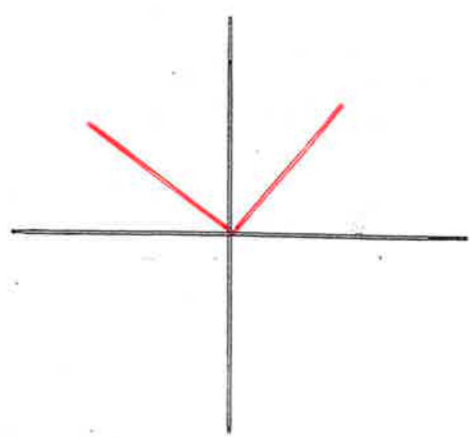
critical numbers are -1 and -3

b)

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

So

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



$f'(x)$ is never 0. The only critical number is 0.

The **absolute maximum or minimum** of a continuous function on a **closed interval** is either a **local maximum or minimum**, in which case it **occurs at a critical number**, or it **occurs at the endpoint** of the interval. This concept gives us the following definition.

Procedure for Finding the Absolute Maximum and Minimum Values of a Continuous Function on a Closed Interval $[a, b]$

1. Find the values of f at the critical numbers of f in (a, b)
2. Find the values of the endpoints; evaluate $f(a)$ and $f(b)$
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Ex 6: Find the absolute maximum and minimum values of the function:

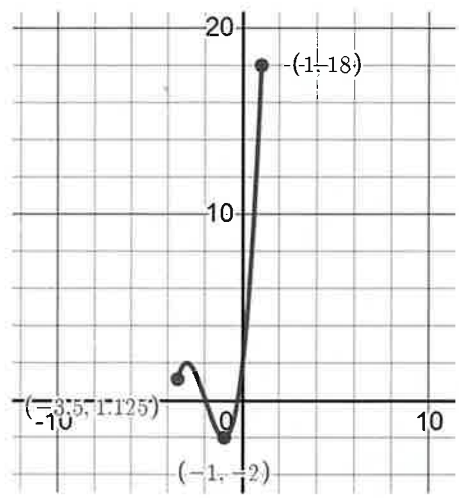
$$f(x) = x^3 + 6x^2 + 9x + 2, -3.5 \leq x \leq 1$$

Find critical numbers

$f'(x) = 3x^2 + 12x + 9$

$= 3(x^2 + 4x + 3)$

$= 3(x+3)(x+1)$



Critical numbers are:

$-3, -1$

Endpoint values are:

-3.5 and 1

$f(-3) = 2$ $f(-1) = -2$

$f(-3.5) = 1.125$ $f(1) = 18$

Absolute maximum: $f(1) = 18$

Absolute minimum: $f(-1) = -2$

Ex 7: Find the absolute maximum and minimum values of the function:

$$g(x) = x^{\frac{2}{3}}(5+x) \qquad -5 \le x \le 1$$

$g(x) = 5x^{2/3} + x^{5/3}$

$g'(x) = \frac{10}{3}x^{-\frac{1}{3}} + \frac{5}{3}x^{2/3}$

$g'(x) = \frac{10}{3x^{\frac{1}{3}}} + \frac{5x^{2/3}}{3} \rightarrow \frac{10+5x}{3x^{\frac{1}{3}}}$

$g'(x) = 0$ when $x = -2$

$g'(x)$ is undefined when $x = 0$

Critical numbers: $-2, 0$

End Points: $-5, 1$

$g(-2) = 5(-2)^{2/3} + (-2)^{5/3}$

$= 5(4)^{1/3} + (-32)^{1/3}$

$\doteq 4.76$

$g(0) = 0$

$g(-5) = 0$

$g(1) = 6$

Absolute Maximum: $g(1) = 6$

Absolute Minimum: $g(0)$ and $g(-5) = 0$

Homework Problems

- Section 4.2: #2ade, 3acde, 4adg, 6