Section 4.1 – Radicals

- We have seen Radicals in PC 11, and we looked at basic transformation in Section 2 of this course.
- This section will explore them at a deeper level.

Definition of a Radical $\sqrt[n]{a}$

Let n **be a positive real number greater than 1**, and let a **be any real number**.

Then:

1. If $a > 0$ **and** n **is even**, we have **two real number solutions**, a **positive and negative** solution
2. If $a < 0$ **and** n **is even**, we have **no real number solution** (Even root of a negative Does Not Exist)
3. If $a > 0$ **and** n **is odd**, we have **one real number solution**, only a **positive one**
4. If $a < 0$ **and** n **is odd**, we have **one real number solution**, only a **negative one**
5. If $a = 0$, $\sqrt[n]{0} = 0$, regardless of n begin odd or even

- A bit of Formal Vocabulary:

Radical Notation

The solution x, is the n^{th} root of a if $x^n = a \ and \ \sqrt[n]{a} = x$

n is called the index of a radical → $\sqrt[n]{a}$ ← a is called the **radicand**

Example 1: Solve the following for x, a real number:

a) $x^2 = 1$	b) $x^3 = -27$	c) $x^4 = 5$	d) $x^4 = -5$	e) $x^5 = 5$	f) $x^5 = -5$

Solution 1:

a)	b)	c)
$x^2 = 1$ $x = \pm\sqrt{1}$ $x = \pm 1$	$x^3 = -27$ $x = \sqrt[3]{-27}$ $x = -3$	$x^4 = 5$ $x = \pm\sqrt[4]{5}$ $x \approx \pm 1.5$
d)	**e)**	**f)**
$x^4 = -5$ $x = \pm\sqrt[4]{-5}$ $x = Does\ Not\ Exist$	$x^5 = 5$ $x = \sqrt[5]{5}$ $x \approx 1.38$	$x^5 = -5$ $x = \sqrt[5]{-5}$ $x \approx -1.38$

Graphing Radicals in the Form $y = a\sqrt{b(x-h)} + k$

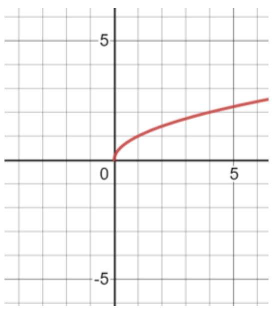
The connection between Transformations stays the same here

- The $a - value$ is a vertical stretch/compression (when $a < 0$, it is a reflection in the $x - axis$)
- The $b - value$ is a horizontal stretch/compression (when $b < 0$, it is a reflection in the $y - axis$)
- The $h - value$ is a horizontal shift left/right
- The $k - valuw$ is a vertical shift up/down

Example 2: Graph a) $y = \sqrt{x}$ b) $y = \sqrt{-x}$ c) $y = -\sqrt{x}$ d) $y = -\sqrt{-x}$

Solution 2:

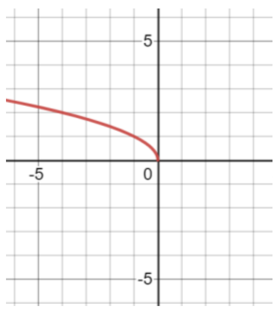
a)



$Domain: x \geq 0$

$Range: y \geq 0$

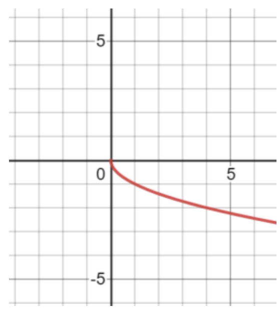
b)



$Domain: x \leq 0$

$Range: y \geq 0$

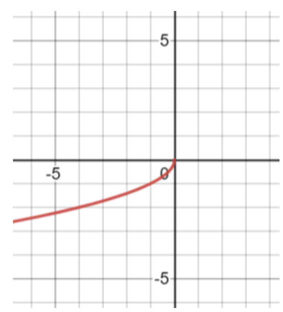
c)



$Domain: x \geq 0$

$Range: y \leq 0$

d)



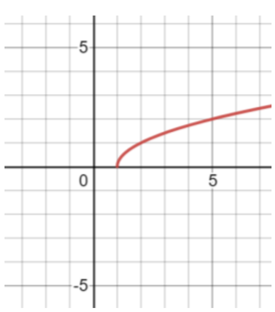
$Domain: x \leq 0$

$Range: y \leq 0$

Example 3: Graph a) $y = \sqrt{x-1}$ b) $y = \sqrt{x+1}$ c) $y = \sqrt{x} + 1$ d) $y = \sqrt{x} - 1$

Solution 3:

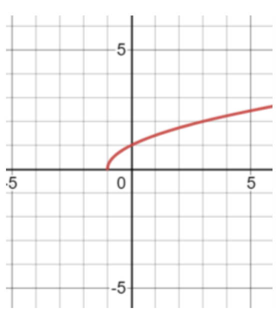
a)



$Domain: x \geq 1$

$Range: y \geq 0$

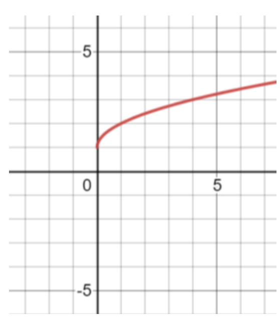
b)



$Domain: x \geq -1$

$Range: y \geq 0$

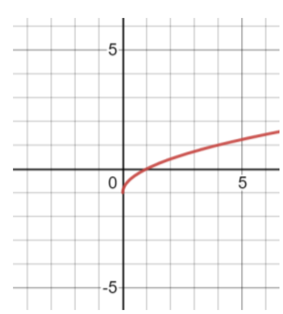
c)



$Domain: x \geq 0$

$Range: y \geq 1$

d)



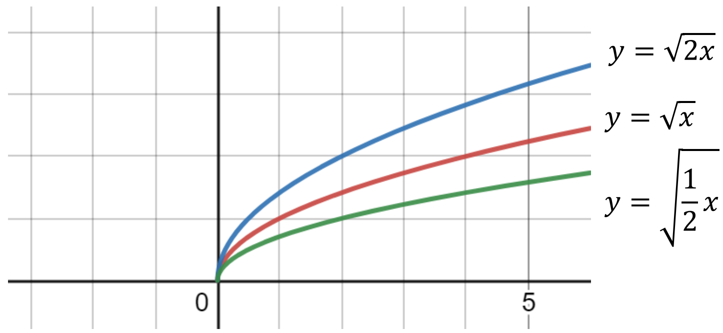
$Domain: x \geq 0$

$Range: y \geq -1$

Example 4: Graph $y = \sqrt{x}$ $y = \sqrt{2x}$ $y = \sqrt{\frac{1}{2}x}$

Solution 4:

Horizontal Compressions are not necessarily intuitive, it is easiest to plug values in for x and track the behaviour of $f(x)$



- When we consider Radicals, we have to consider the Domain and the Range
- There are values of x that are not allowed. There are vertical asymptotes and horizontal asymptotes that terminates the radical, consider this when we have Horizontal and Vertical Translations

Example 5: Graph a) $y = 2\sqrt{4 - x} + 1$ b) $y = -\sqrt{2x - 4} - 1$

Solution 5:

a) **Domain:** $4 - x \geq 0 \quad \rightarrow \quad \boldsymbol{x \geq 4}$

Range: Since $2\sqrt{4-x} \geq 0$ then $2\sqrt{4-x} + 1$ is a Vertical Translation of $+1$ so:

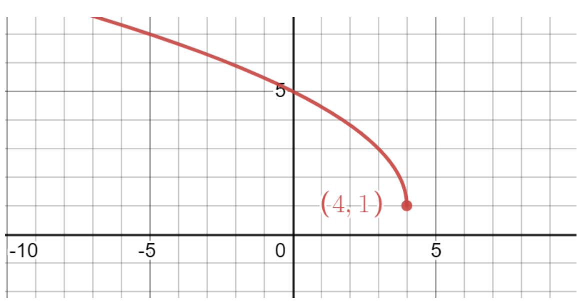
$$\boldsymbol{y \geq 1}$$

By Transformation:
$y = \sqrt{x} \rightarrow y = 2\sqrt{4 - x} + 1 \rightarrow y = 2\sqrt{-(x-4)} + 1$

$$\boldsymbol{(a, b) \rightarrow (-a + 4, 2b + 1)}$$

$$(0,0) \rightarrow (4,1); \quad (1,1) \rightarrow (3,3)$$

$$(4,2) \rightarrow (0,5); \quad (9,3) \rightarrow (-5,7)$$



b) **Domain:** $2x - 4 \geq 0 \rightarrow 2x \geq 4 \rightarrow \boldsymbol{x \geq 2}$

Range: Since $-\sqrt{2x-4} \leq 0$ then $-\sqrt{2x-4} - 1$ is a Vertical Translation of -1 so:

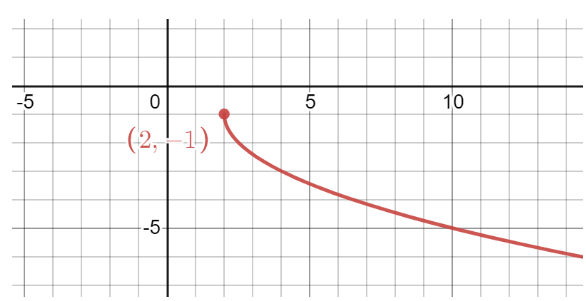
$$\boldsymbol{y \leq -1}$$

By Transformation:
$y = \sqrt{x} \rightarrow y = -\sqrt{2x - 4} - 1 \rightarrow y = -\sqrt{2(x-2)} - 1$

$$\boldsymbol{(a, b) \rightarrow (\frac{1}{2}a + 2, -b - 1)}$$

$$(0,0) \rightarrow (2, -1); \quad (1,1) \rightarrow (2.5, -2)$$

$$(4,2) \rightarrow (4, -3); \quad (9,3) \rightarrow (6.5, -4)$$



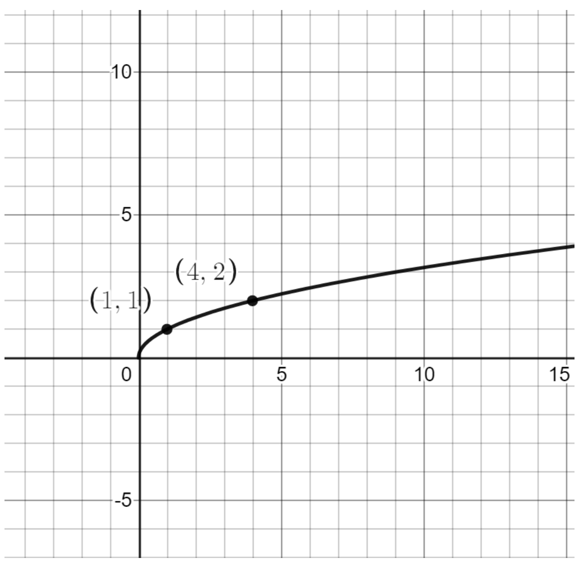
Graphing Radical Functions with Even and Odd Root Indexes

- As we saw with **even root indexes**, we have **domain restrictions** when the **radicand is negative**
- If the **root index if odd**, there are **no domain restrictions**, negative radicands work too
- Let's see the difference graphically.

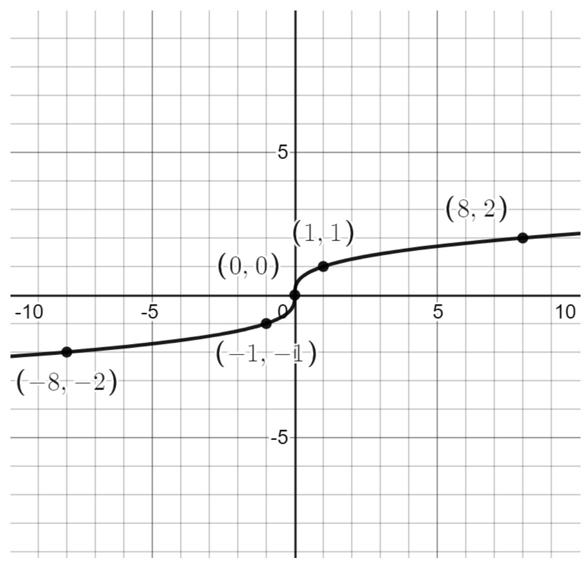
Example 6: How do the graphs of: a) $y=\sqrt{x}$ b) $y=\sqrt[3]{x}$ c) $y=\sqrt[4]{\sqrt{x}}$ d) $y=\sqrt[5]{x}$, differ?

Solution 6:

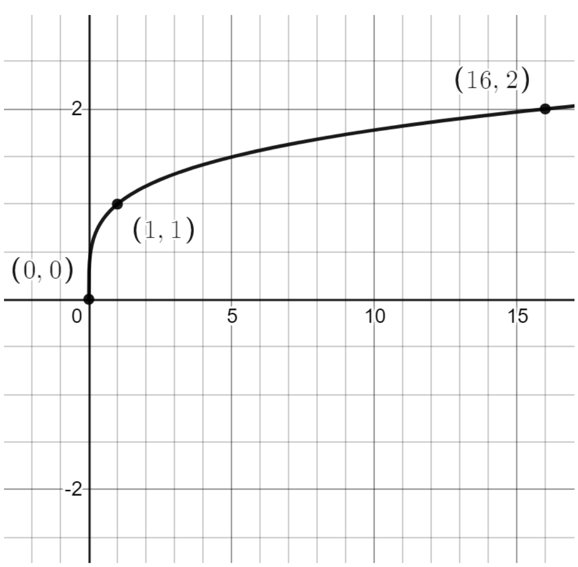
a) $y=\sqrt{x}$ $\quad Domain\colon\ x\geq 0,\ \ Range\colon\ y\geq 0$



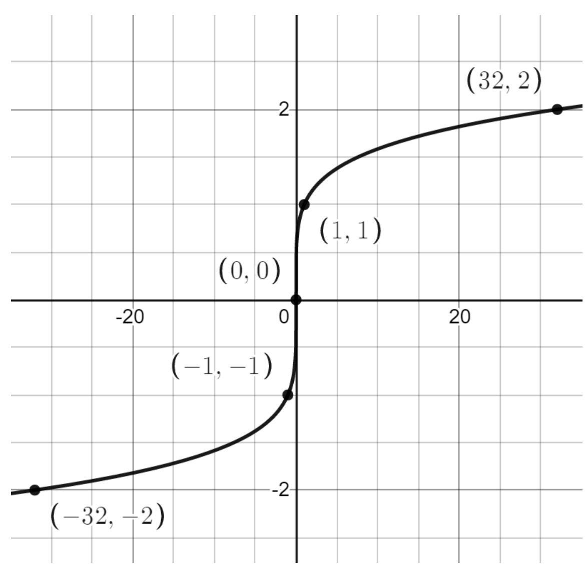
b) $y=\sqrt[3]{x}$ $\quad Domain\ and\ Range\colon All\ Real\ Numbers$



c) $y=\sqrt[4]{x}$ $\quad Domain\colon\ x\geq 0,\ \ Range\colon\ y\geq 0$



d) $y=\sqrt[5]{x}$ $\quad Domain\ and\ Range\colon All\ Real\ Numbers$

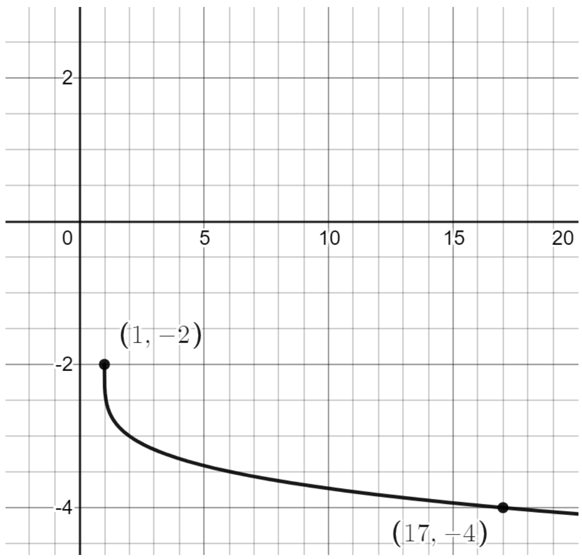


Example 7: Graph a) $y = -\sqrt[4]{x-1} - 2$ and b) $y = -\sqrt[3]{x-1} - 2$

Solution 7:

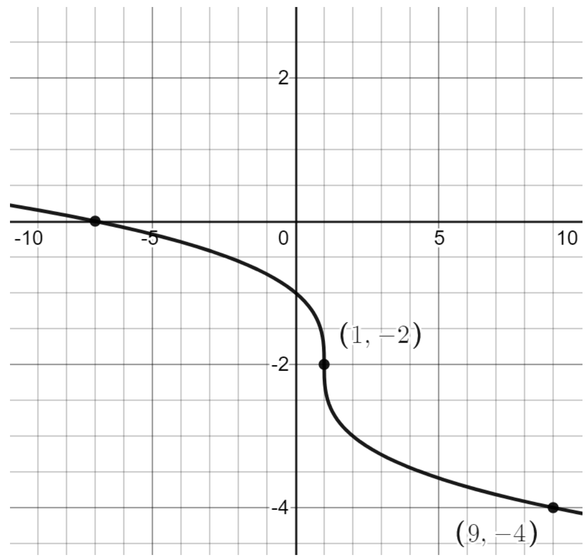
a) $y = -\sqrt[4]{x-1} - 2$

$Domain: \ x \geq 1, \quad Range: \ y \leq -2$



b) $y = -\sqrt[3]{x-1} - 2$

$Domain \ and \ Range: All \ Real \ Numbers$



Graphing $y = f(x)$ and $y = \sqrt{f(x)}$

- Due to Domain Restrictions on $y = \sqrt{f(x)}$ and all other even root functions, graphs look different

Example 8: Graph $y = x^2$ and $y = \sqrt{x^2}$

Solution 8:

For $y = x^2$:

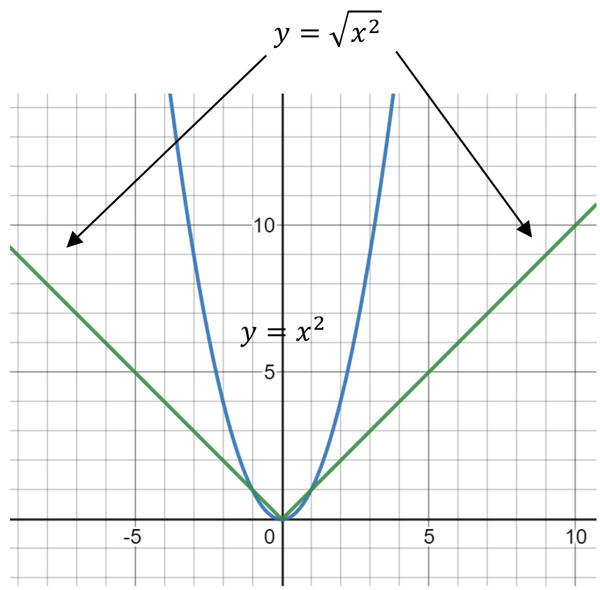
Domain: All Real Numbers

Range: $y \geq 0$

For $y = \sqrt{x^2}$

Domain: All Real Numbers

Range: $y \geq 0$



Example 9: Graph $y = x^3$ and $y = \sqrt{x^3}$

Solution 9:

For $y = x^3$:

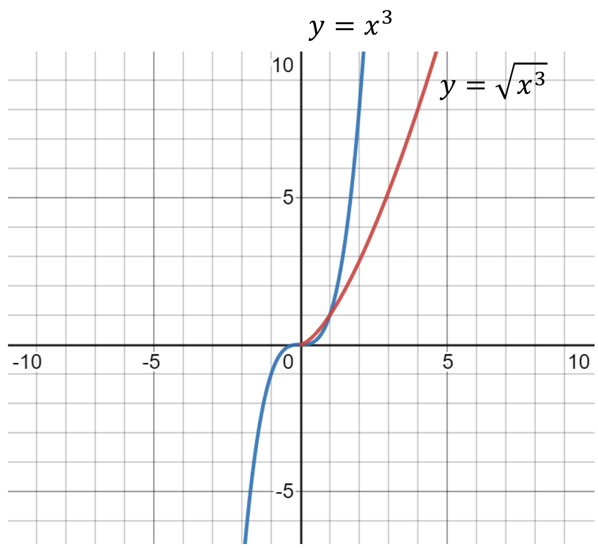
Domain: All Real Numbers

Range: All Real Numbers

For $y = \sqrt{x^3}$

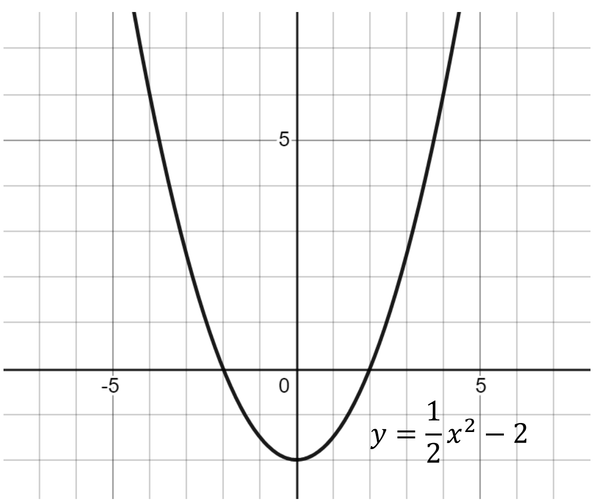
Domain: $x \geq 0$

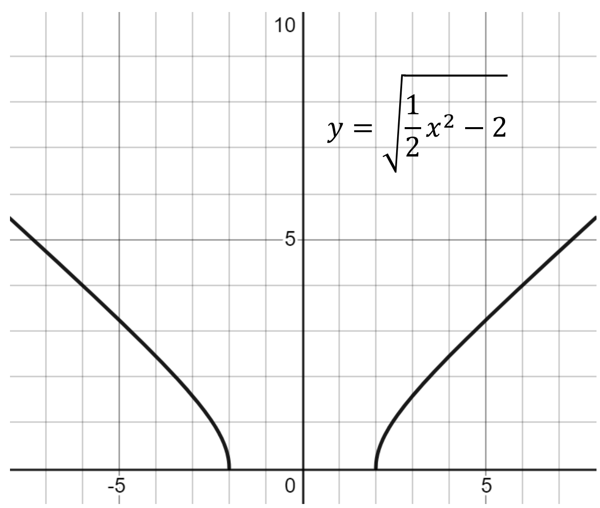
Range: $y \geq 0$



Example 10: Graph $f(x) = \frac{1}{2}x^2 - 2$ and $y = \sqrt{f(x)}$

Solution 10: Radicals have Domain Restrictions, which is why the graphs vary so much





For $y = \frac{1}{2}x^2 - 2$

Domain: All Real Numbers

Range: $y \geq 2$

For $y = \sqrt{\frac{1}{2}x^2 - 2}$

Domain: $\frac{1}{2}x^2 - 2 \geq 0$

$\frac{1}{2}x^2 \geq 2 \quad \rightarrow \quad x^2 \geq 4$

$x \geq 2 \quad and \quad x \leq -2$

For $y = \sqrt{\frac{1}{2}x^2 - 2}$

Range: $y \geq 0$

Section 4.1 – Practice Problems

1. Answer the following questions to lockdown your vocabulary.

a) In radical notation, $\sqrt[n]{x}, x$ is called what?	b) In radical notation, what do we call the root symbol?
c) In radical notation, $\sqrt[n]{x}, n$ is called what?	d) The n^{th} root of x is written?
e) $\sqrt{25} = 5$ is read the ________ root of 25 equals 5.	f) $\sqrt[3]{-27} = -3$ is read the ________ root of -27 equals -3.
g) The n^{th} root of x is not a real number if n is ____________ and x is __________	h) The Domain of a radical with even index excludes all values that make the radicand ____________

2. Solve for x.

a) $x^2 = 9$	b) $x^2 = -9$	c) $x^3 = 8$	d) $x^3 = -8$
e) $x^4 = 1$	f) $x^4 = -1$	g) $x^5 = 32$	h) $x^5 = -32$

3. Simplify each radical

a) $\sqrt{4x^2},\ x \geq 0$	b) $\sqrt{4x^2},\ x < 0$
c) $\sqrt[3]{27x^3},\ x < 0$	d) $\sqrt[3]{-27x^3},\ x \geq 0$

4. What is the Domain and Range of the following functions?

a) $y = x$

b) $y = \sqrt{x}$

c) $y = \sqrt{1 - x}$

d) $y = -\sqrt{x - 1}$

e) $y + 2 = \sqrt{1 - x}$

f) $y - 2 = \sqrt{x - 1}$

g) $y+3=\sqrt{2x-4}$

h) $y-3=-\sqrt{2x+4}$

i) $y=-\sqrt{-2x-4}+3$

j) $y=\sqrt{x^2-4}$

k) $y=-\sqrt{4-x^2}$

l) $y=-\sqrt{x^3-8}$

5. Match the equation with the graph.

a) $f(x) = \sqrt{-x}$	
b) $f(x) = -\sqrt{x}$	
c) $f(x) = \sqrt[3]{x}$	
d) $f(x) = \sqrt{1-x}$	
e) $f(x) = \sqrt{x^2}$	
f) $f(x) = -\sqrt{x-1}$	
g) $f(x) = -\sqrt{x} - 1$	
h) $f(x) = 1 - \sqrt[3]{-x}$	
i) $f(x) = 1 - \sqrt{x-1}$	
j) $f(x) = 1 - \sqrt{1-x}$	
k) $f(x) = \sqrt{x^2-1}$	
l) $f(x) = 1 + \sqrt[3]{-x}$	
m) $f(x) = -1 - \sqrt{1-x}$	
n) $f(x) = -1 + \sqrt{x+1}$	

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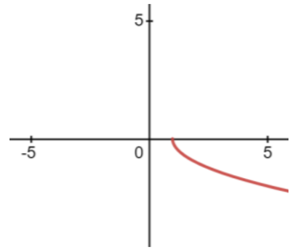

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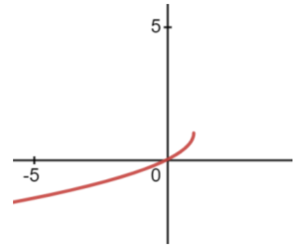

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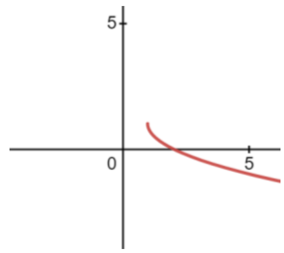

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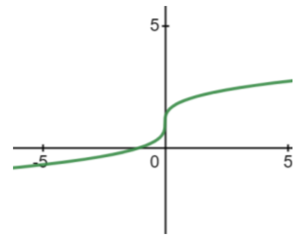

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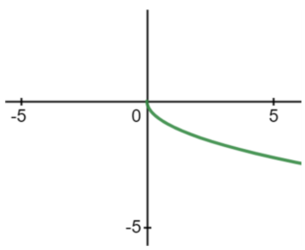

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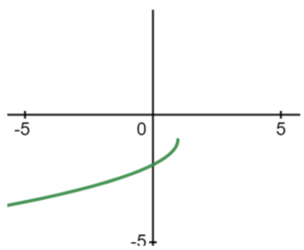

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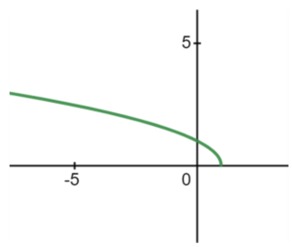

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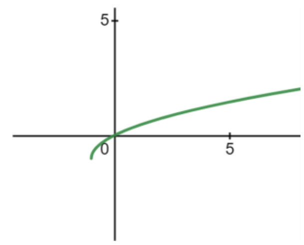

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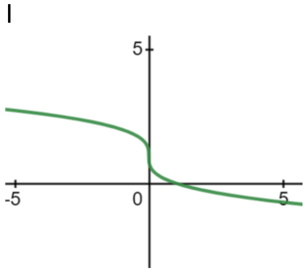

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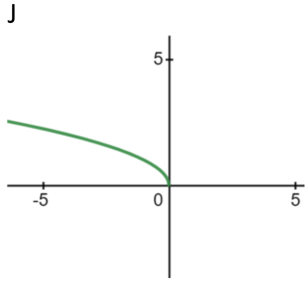

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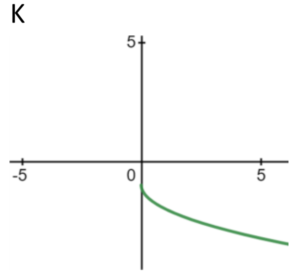

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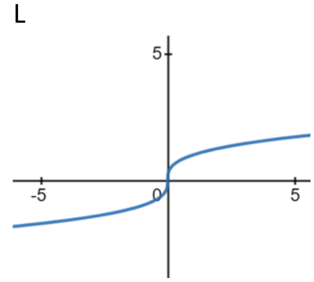

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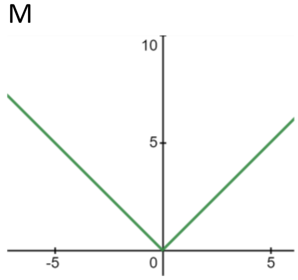

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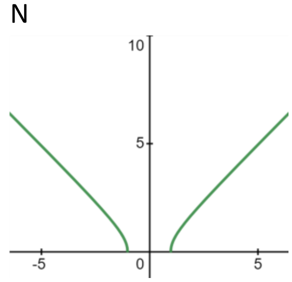
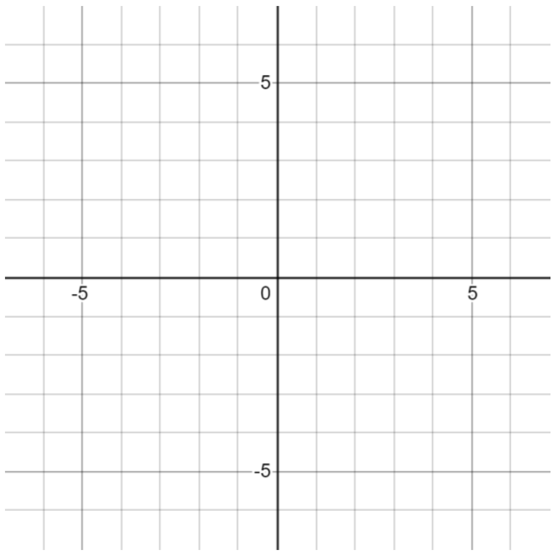


6. Graph the following functions. State the Domain and the Range

a) $f(x) = 2x$

Domain:

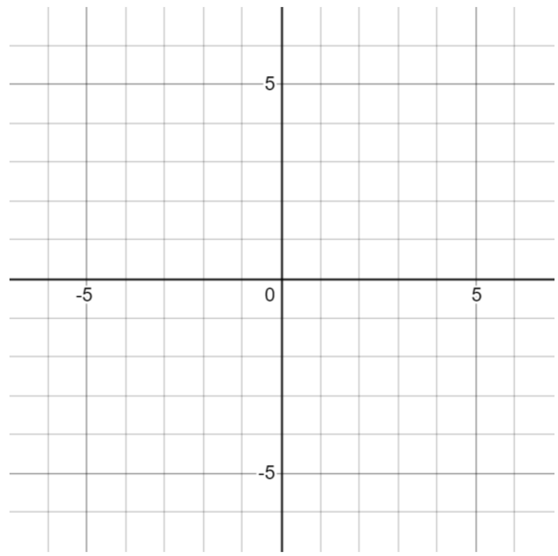
Range:



b) $f(x) = \sqrt{2x}$

Domain:

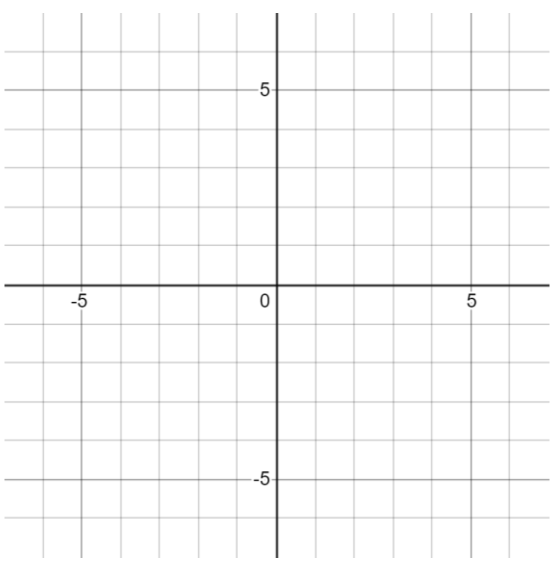
Range:



c) $f(x) = 4 - x^2$

Domain:

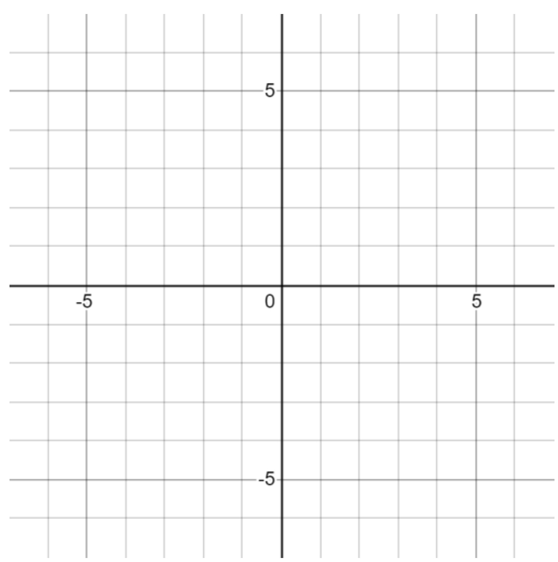
Range:



d) $f(x) = \sqrt{4 - x^2}$

Domain:

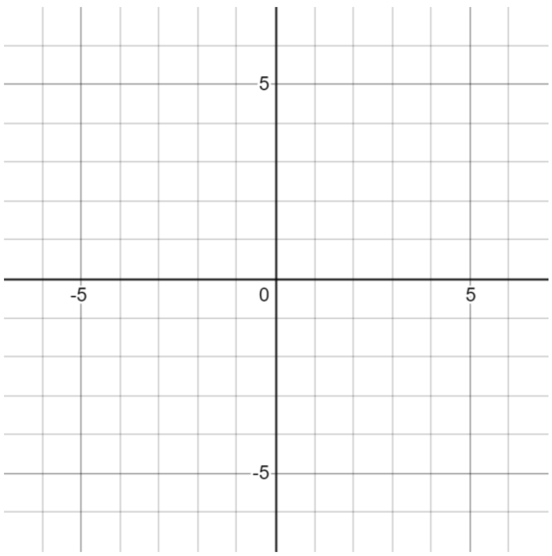
Range:



e) $f(x) = \frac{1}{3}x^2 - 3$

Domain:

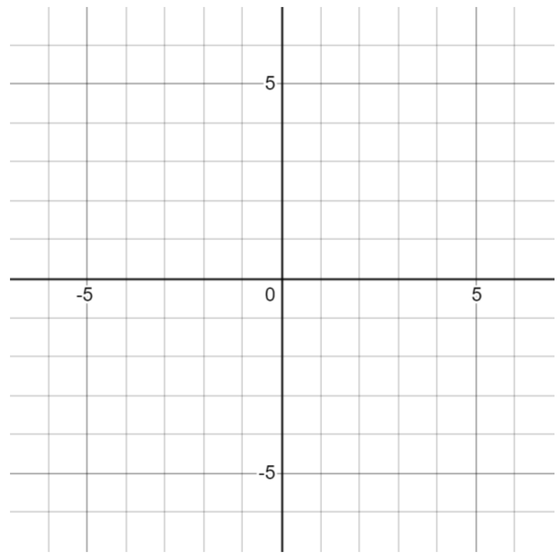
Range:



f) $f(x) = \sqrt{\frac{1}{3}x^2 - 3}$

Domain:

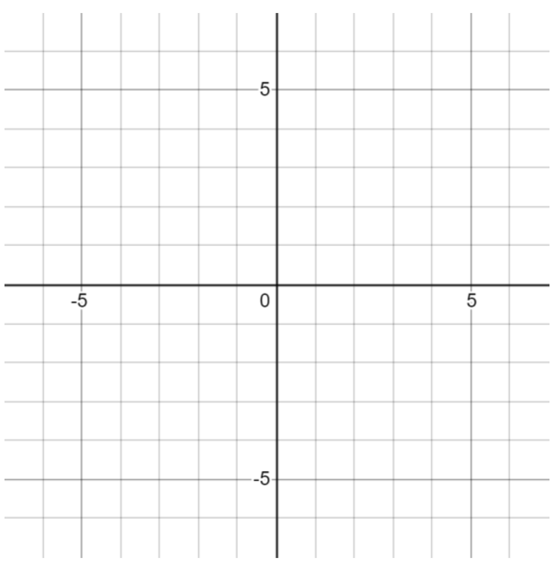
Range:



g) $f(x) = -\frac{1}{8}x^3 + 1$

Domain:

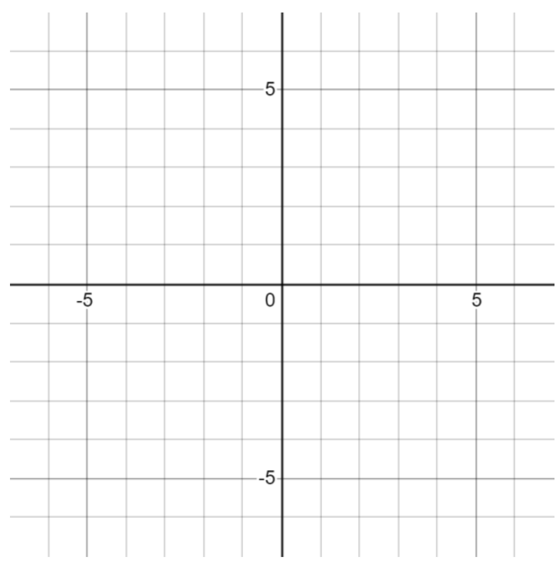
Range:



h) $f(x) = \sqrt{-\frac{1}{8}x^3 + 1}$

Domain:

Range:



See Website for Detailed Answer Key of the Remainder of the Questions

Extra Work Space