Section 3.7 – Rational Functions

This booklet belongs to:______________________________Block: ______

- A function f is a Rational Function if:

$$\boldsymbol{f(x) = \frac{g(x)}{h(x)}}$$

Where $h(x)$ *and* $g(x)$ are polynomials and $h(x)$ does not equal 0.

- In this section we will look at the properties of Rational Functions and how they are graphed
- In the section that follows we will explore the mathematical operations involving Rational Functions

- The **DOMAIN** of a Rational function consists of all Real Numbers, **except** when it results in **a zero denominator**

For example:

$g(x) = \frac{1}{x+2}$ *has a Domain of all Real Numbers except when* $x = -2$; *this is denoted* $x \neq -2$

$h(x) = \frac{x}{x^2 - 9}$ *has a Domain of all Real Numbers except when* $x = -3, 3$; *this is denoted* $x \neq \pm 3$

$j(x) = \frac{x-1}{x^2+1}$ *has a Domain of all Real Numbers*

Graphing a Rational Function

Consider the rational function $f(x) = \frac{1}{x}$ The Function is Not Defined when $x = 0$.

- It is important to consider the behaviour of the graph as it approaches this undefined point
- Some values for $f(x)$ as x approaches 0 are:

x	$f(x)$
1	1
0.1	10
0.01	100
0.001	1000

x	$f(x)$
-1	-1
-0.1	-10
-0.01	-100
-0.001	-1000

This pattern is written:

As $x \to 0^+, f(x) \to +\infty$

(As x approaches 0 from the **right**, $f(x)$ approaches an infinitely **large positive number**)

As $x \to 0^-, f(x) \to -\infty$

(As x approaches 0 from the **left**, $f(x)$ approaches an infinitely **large negative number**)

This behaviour is defined by the **vertical line $x = 0$** and is called a **Vertical Asymptote of the function.**

- It is also important to consider the behaviour of the graph as it moves away from this point
- Some values for $f(x)$ as x approaches $\pm\infty$ are

x	$f(x)$
10	1
100	10
1000	100

x	$f(x)$
-10	-1
-100	-10
-1000	-100

This pattern is written:

As $x \to +\infty, f(x) \to 0^+$

(As x approaches an infinitely **large positive number,** $f(x)$ approaches a very small positive number)

As $x \to -\infty, f(x) \to 0^-$

(As x approaches an infinitely **large negative number,** $f(x)$ approaches a very small negative number)

This behaviour is defined by the **horizontal line $y = 0$** and is called a **Horizontal Asymptote of the function.**

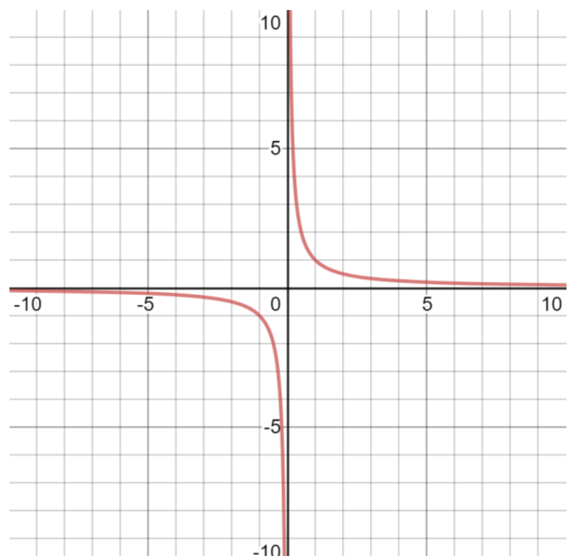
The graph of $f(x) = \frac{1}{x}$

The lines approach $y = 0$ but never cross

Horizontal Asymptote

The lines approach $x = 0$ but never cross

Vertical Asympptote



Example 1: Graph the Rational function $f(x) = \frac{1}{x-1} - 1$

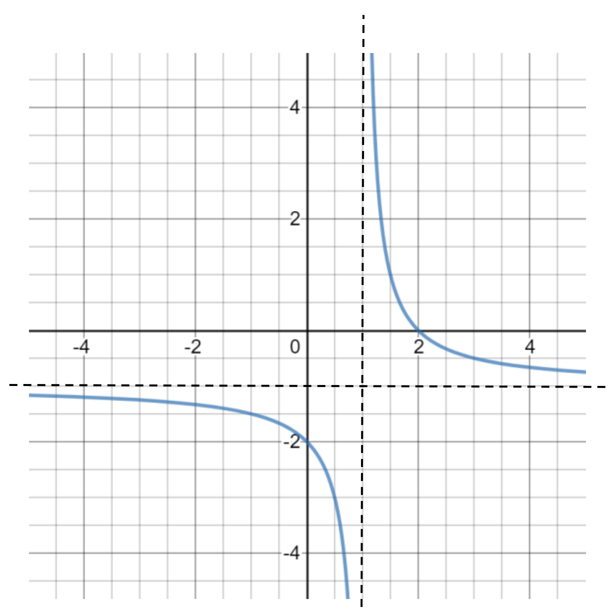
Solution 1: The Domain is $x \neq 1$, so we have the **vertical asymptote.**

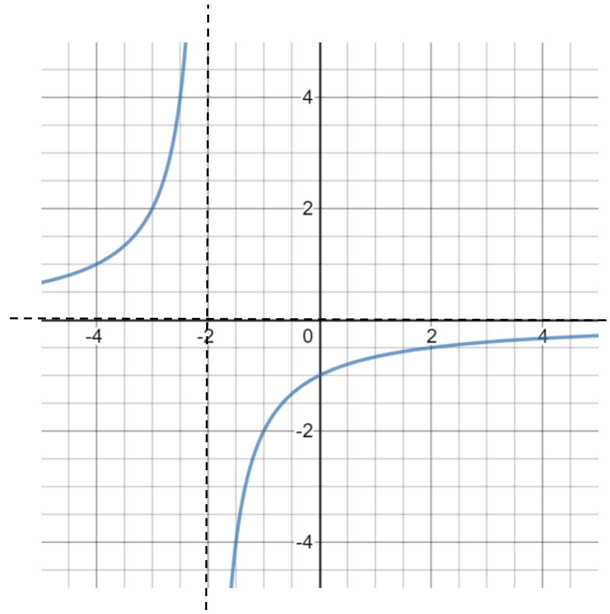
The graph is shifted down 1 unit, so the **horizontal asymptote** is $y = -1$

x	2	0	1^-	1^+	-1000	1000
$f(x)$	0	-2	$-\infty$	∞	-1^-	-1^+

Note:
1^- means slightly less than 1

1^+ means slightly more than 1





Example 2: Graph the Rational function $g(x) = \frac{-2}{x+2}$

Solution 2: The Domain is $x \neq -2$, so we have the **vertical asymptote.**

The graph is not shifted up or down, so the **horizontal asymptote** is $y = 0$

x	0	-2^-	-2^+	-100	100
$g(x)$	-1	∞	$-\infty$	0^+	0^-

Asymptotes

- An asymptote is **not part of the graph**, they represent **points of approach**.
- **Vertical Asymptotes** can **never be crossed**, it is the vertical line representing the **boundary the graph** approaches **as the denominator of the function approaches zero**
- **Horizontal Asymptotes** represent the **boundary the graph approaches as x approaches** $\pm\infty$
- It is possible a graph crosses a horizontal asymptote, but not at the extremes of x

Vertical Asymptotes

- The vertical asymptote is found when the denominator is zero, you may need to factor the denominator to determine this (these) points.

Examples:

$$f(x) = \frac{1}{x+2} \quad \textit{has a Vertical Asymptote } x = -2, \textit{because } x + 2 = 0 \quad \rightarrow \quad x = -2$$

$$h(x) = \frac{1}{x^2 - 4} \quad \textit{has a Vertical Asymptotes } x = 2 \textit{ and } x = -2, \textit{because } x^2 - 4 = 0 \quad \rightarrow \quad x = \pm 2$$

$$j(x) = \frac{x-3}{x^2+1} \quad \textit{has no Vertical Asymptote, since } x^2 + 1 = 0 \quad \rightarrow \quad x = \emptyset$$

Horizontal Asymptotes

- The horizontal asymptote is the value y approaches as $x \to \pm\infty$
- Consider the function

$$\boldsymbol{f(x) = \frac{g(x)}{h(x)}}$$

1. If $h(x)$ is a higher degree than $g(x)$, the Horizontal Asymptote is: $y = 0$
2. If $g(x)$ and $h(x)$ have the same degree, the Horizontal Asymptote is: $y = \frac{leading\ coefficient}{leading\ coefficient}$
3. If $g(x)$ is a higher degree than $h(x)$, there is no Horizontal Asymptote

Examples:

$f(x) = \frac{3x+1}{x-2}$ $\quad has\ a\ Horizontal\ Asymptote\ y = \frac{3}{1} = 3$, the numerator/denominator have the same degree

$h(x) = \frac{2x}{x^2-4}$ $\quad has\ a\ Horizontal\ Asymptote\ y = 0$

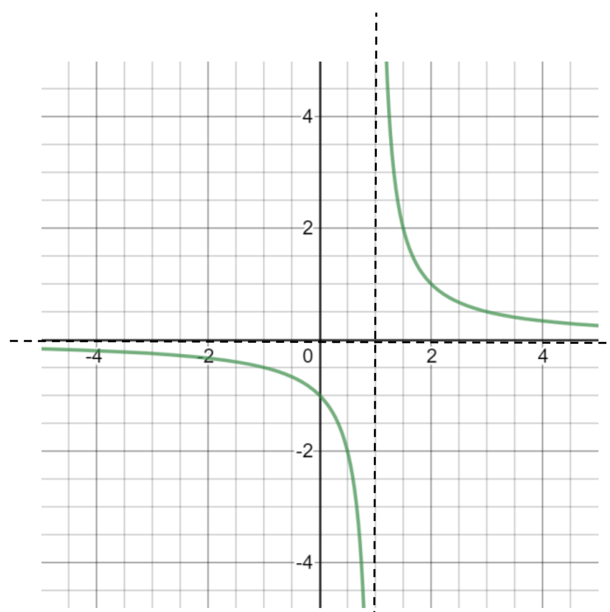
$j(x) = \frac{x^2-3}{x+1}$ $\quad has\ no\ Horizontal\ Asymptote, since\ numerator\ degree\ is\ larger\ than\ the\ denominator$

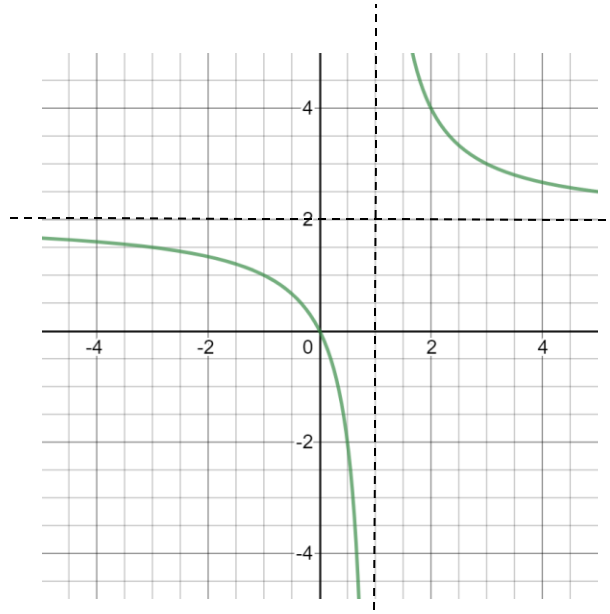
Example 3: Graph the Rational function $g(x) = \frac{1}{x-1}$

Solution 3: **Vertical asymptote is:** $\boldsymbol{x = 1}$

Horizontal asymptote is: $\boldsymbol{y = 0}$

x	0	0.9	1.1	-10	10
$g(x)$	-1	-10	10	-0.1	0.1



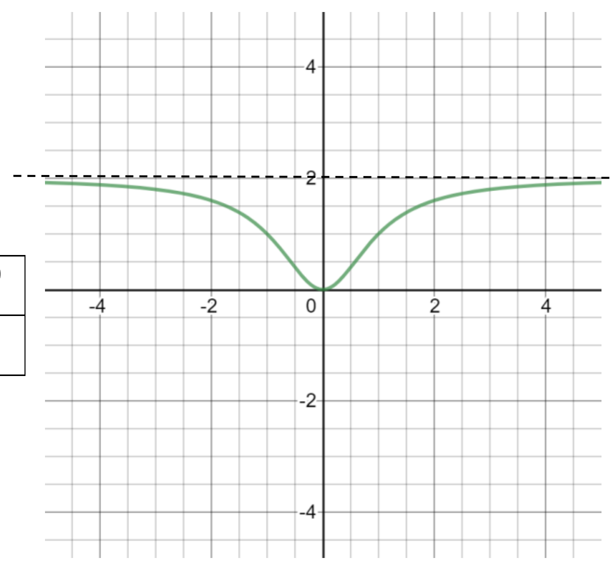


Example 4: Graph the Rational function $g(x) = \frac{2x}{x-1}$

Solution 4: **Vertical asymptote is: $x = 1$**

Horizontal asymptote is: $y = 2$

x	0	0.9	1.1	-50	50
$g(x)$	0	-18	22	1.96	2.04



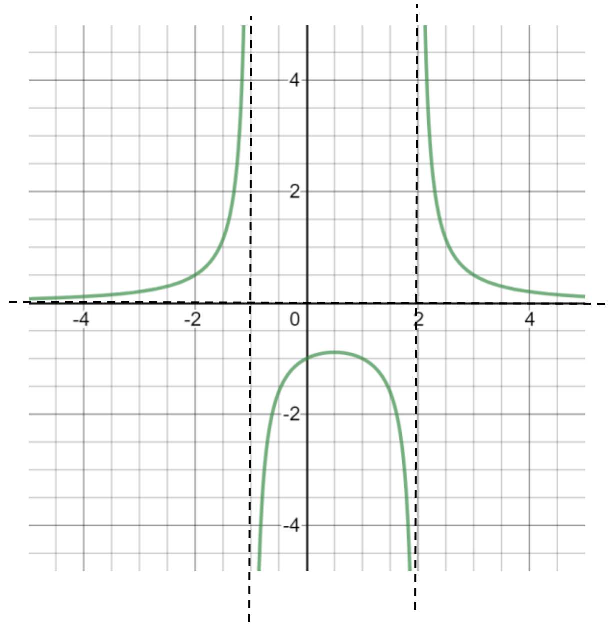
Example 5: Graph the Rational function $g(x) = \frac{2x^2}{x^2+1}$

Solution 5: **Vertical asymptote is: $None$**

Horizontal asymptote is: $y = 2$

x	0	1	-1	-5	5	-100	100
$g(x)$	0	1	1	1.9	1.9	2^-	2^-

Example 6: Graph the Rational function $k(x) = \frac{2}{x^2-x-2}$



Solution 6: $k(x) = \frac{2}{(x-2)(x+1)}$

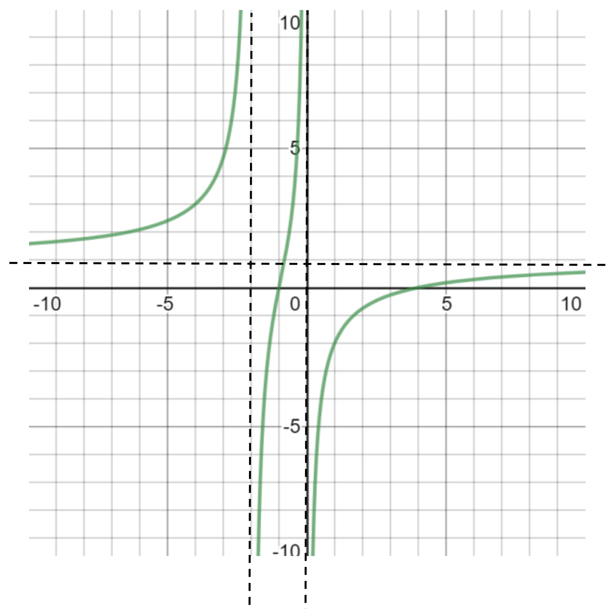
Vertical asymptote is: $\boldsymbol{x = 2, -1}$

Horizontal asymptote is: $\boldsymbol{y = 0}$

x	0	-0.9	-1.1	1.9	2.1	-10	10
$k(x)$	-1	-6.9	6.5	-6.9	6.5	0.02	0.02

- Remember the Horizontal Asymptote is only apparent at the extremes of positive and negative infinity on the $x - axis$

Example 7: Graph the Rational function $f(x) = \frac{x^2-3x-4}{x^2+2x}$



Solution 7: $f(x) = \frac{x^2-3x-4}{x^2+2x} = \frac{(x-4)(x+1)}{x(x+2)}$

Vertical asymptote is: $\boldsymbol{x = 0, -2}$

Horizontal asymptote is: $\boldsymbol{y = 1}$

$x - intercept = (-1, 0)\ and\ (4, 0)$

$y - intercept = (no\ y - intercept)$

x	-0.1	0.1	-2.1	-1.9	100	-100
$f(x)$	∞	$-\infty$	∞	$-\infty$	1^-	1^+

- You can see the graph crosses the Horizontal Asymptote, but it doesn't matter, Horizontal Asymptotes are only at the extremes of positive and negative infinity on the $x - axis$

Section 3.7 – Practice Problems

Find the $x\ and\ y\ intercepts$ of the following rational functions

1. $y = \dfrac{x+4}{x-1}$

2. $y = \dfrac{x}{x+3}$

3. $y = \dfrac{(x-1)(x+4)}{(x+2)(x-2)}$

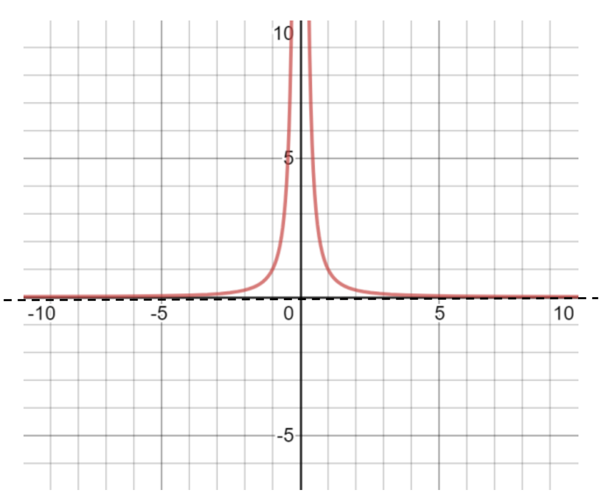
4. $y = \dfrac{3}{x+2}$

5. $y = \dfrac{1}{x^2+1}$

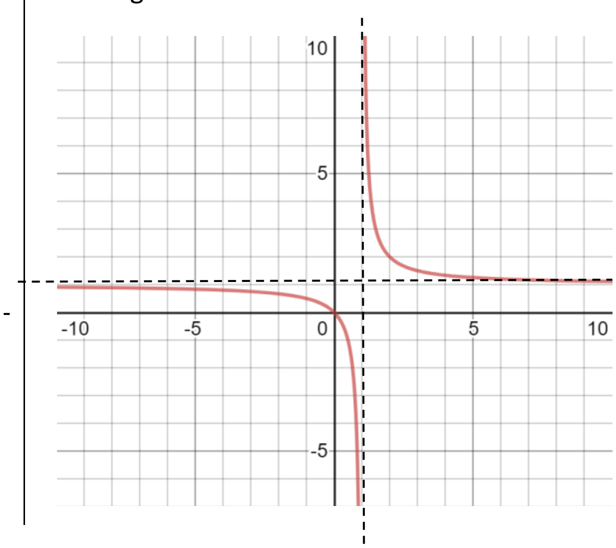
6. $y = \dfrac{1}{x^2-4}$

Determine the Domain and Range of the following graphs

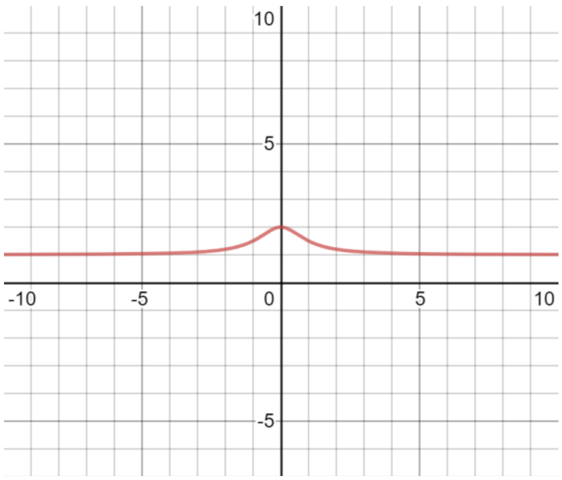
7. Domain:
   Range:



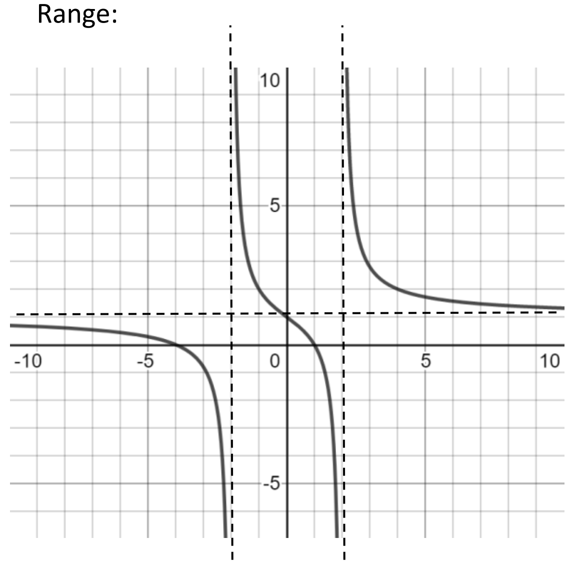
8. Domain:
   Range:



9. Domain:
Range:



10. Domain:
Range:



Find all the Vertical and Horizontal Asymptotes

11. $f(x) = \dfrac{2}{x+3}$

12. $g(x) = \dfrac{x+4}{x+2}$

13. $h(x) = \dfrac{2}{x} - 2$

14. $t(x) = \dfrac{1}{x+2} - 1$

15. $j(x) = \dfrac{2x^2 - 2}{x^2 - 4}$

16. $k(x) = \dfrac{3x + 4}{2x - 5}$

17. $l(x) = \dfrac{x - 1}{x^2 - 1}$

18. $m(x) = \dfrac{-3x^2 - 3x + 6}{x^2 - 9}$

19. $n(x) = \dfrac{x^2 - 3x - 4}{x^2 + x - 6}$

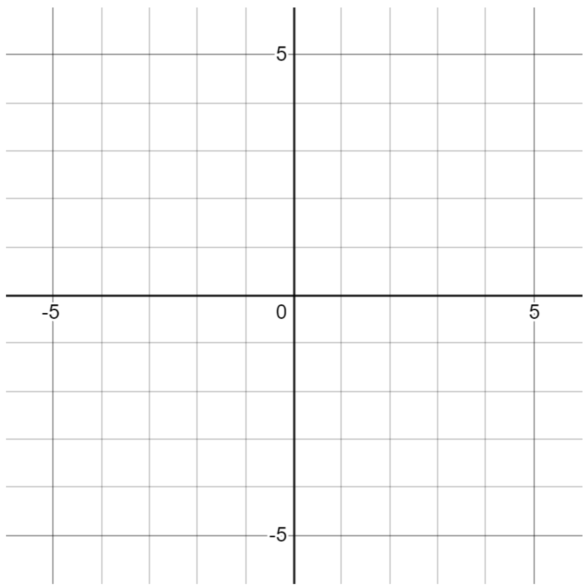
20. $r(x) = \dfrac{1}{(x - 1)^2}$

21. $p(x) = \dfrac{x^2 - 4}{x^2 + 2x - 3}$

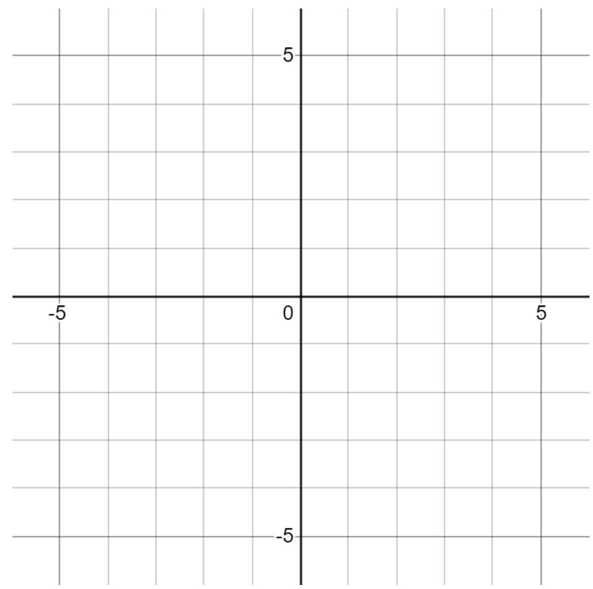
22. $q(x) = \dfrac{x^2 - 4x + 3}{2x^2 - 8x}$

Sketch the graph of the following functions. Label the asymptotes.

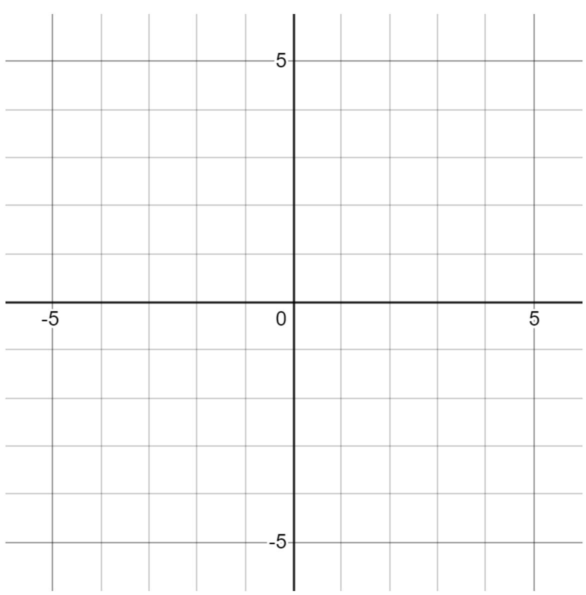
23. $p(x) = -\frac{1}{x^2}$



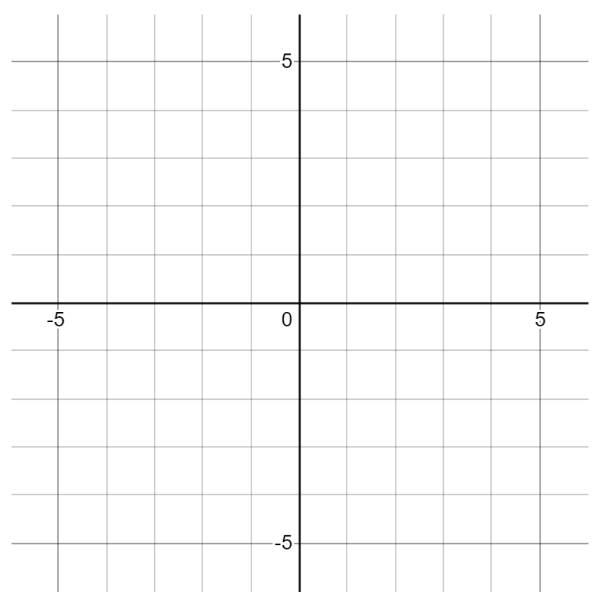
24. $f(x) = \frac{1}{x-2}$



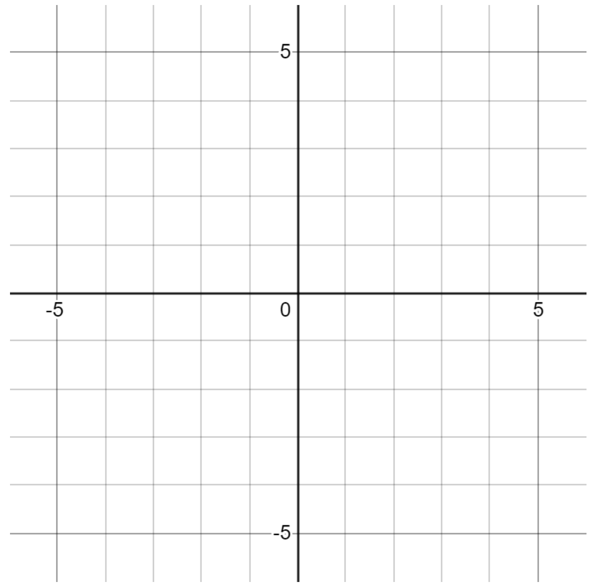
25. $g(x) = \frac{1}{x+2} - 1$



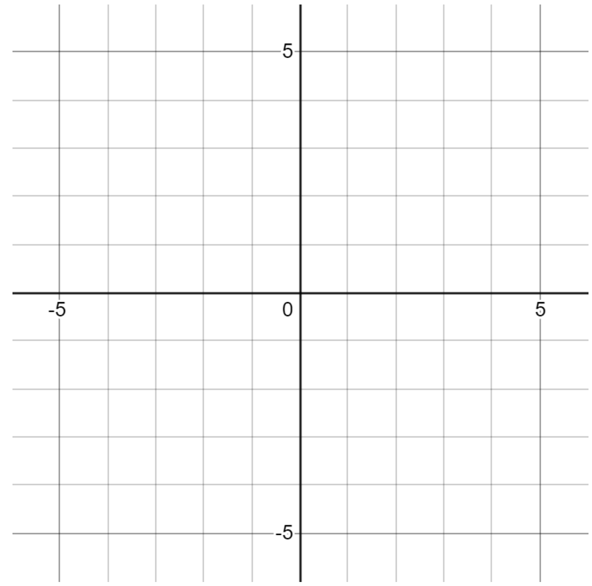
26. $k(x) = \frac{x}{x^2 - 4}$



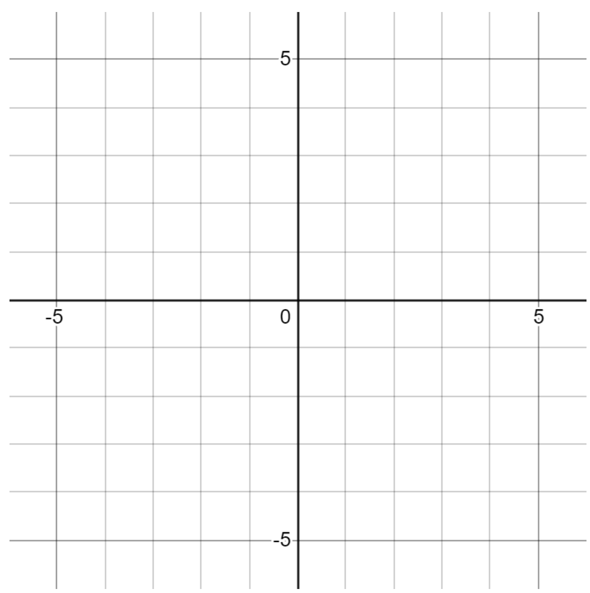
27. $m(x) = \dfrac{x^2 - 1}{x^2 - 4}$



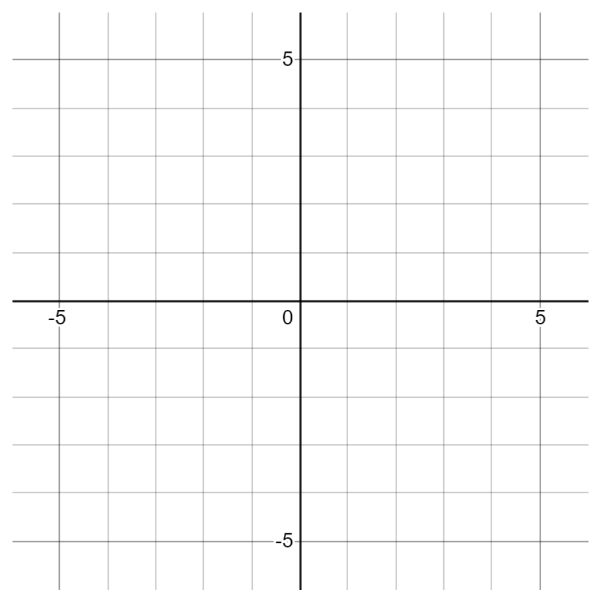
28. $j(x) = \dfrac{x^2 - 4}{x^2 + 1}$



29. $s(x) = \dfrac{x + 2}{x^2 + x - 6}$



30. $t(x) = \dfrac{x - 1}{x^3 - 4x}$



Answer Key – Section 3.7

1. $(-4,0),(0,-4)$	15. $x=-2,2, y=2$
2. $(0,0)$	16. $x=\frac{5}{2}, y=\frac{3}{2}$
3. $(-4,0),(1,0),(0,1)$	17. $x=-1,1, y=0$
4. $(0,\frac{3}{2})$, $No\ x-int$	18. $x=-3,3, y=-3$
5. $(0,1)$, $No\ x-int$	19. $x=-3,2, y=1$
6. $(0,-\frac{1}{4})$, $No\ x-int$	20. $x=1, y=0$
7. $D: x\neq 0$ $R: y\geq 0$	21. $x=-3,1, y=1$
8. $D: x\neq 1$ $R: y\neq 1$	22. $x=0,4, y=\frac{1}{2}$
9. $D: All\ Real\ Numbers$ $R: 1<y\leq 2$	23. $See\ Website$
10. $D: x\neq 2,-2$ $R: All\ Real\ Nmbers$	24. $See\ Website$
11. $x=-3, y=0$	25. $See\ Website$
12. $x=-2, y=1$	26. $See\ Website$
13. $x=0, y=-2$	27. $See\ Website$
14. $x=-2, y=-1$	28. $See\ Website$
	29. $See\ Website$
	30. $See\ Website$

Extra Work Space