$\frac{dV}{dt}$ units cm^3/min $\frac{dV}{dr}$ units cm^3/cm

3.5 Related Rates

Problems involving related rates require the use of the Chain Rule. We are given the rate of change of one variable with respect to another and then using the Chain Rule we can calculate the rate of change of another related quantity. You must use a governing equation for the problem then differentiate both sides of the equation and solve for the rate you are interested in knowing.

volume with respect to time

Ex. 1

A spherical snowball is melting in such a way that its volume is *decreasing* at a rate of 1 cm^3/min. At what rate is the radius decreasing when the radius is 5 cm.

Let Volume of snowball sphere be given by $V = \frac{4}{3}\pi r^3$, thus $\boxed{\frac{dV}{dr} = 4\pi r^2}$

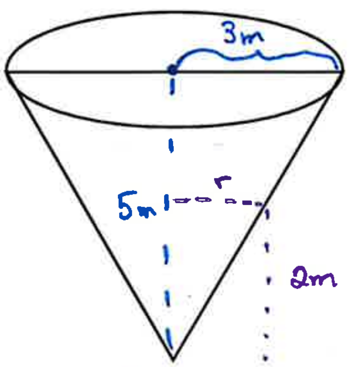
$\frac{dV}{dt} = -1\, cm^3/min$ Need $\frac{dr}{dt}$ ← consider units cm/min

From the Chain Rule: $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \rightarrow -1 = 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-1}{4\pi r^2}$ at $r = 5$ $\frac{dr}{dt} = \frac{1}{4\pi (5)^2} = \frac{1}{100\pi} = \boxed{0.003\, cm/min}$

Ex. 2

A water tank is built in the shape of a circular cone with height 5 m and diameter 6 m at the top. Water is being pumped into the tank at a rate of 1.6 m^3/min. Find the rate at which the water level is rising when the water is 2 m deep.



$\frac{dV}{dt} = 1.6\, m^3/min$ radius is 3m

Volume of cone: $\frac{1}{3}\pi r^2 h$

Need $\frac{dh}{dt}$ at $h = 2$

Also consider similar triangles ratios of radius to height

$\frac{r}{h} = \frac{3}{5}$ so $\boxed{r = \frac{3h}{5}}$

$V = \frac{1}{3}\pi \left(\frac{3h}{5}\right)^2 h = \frac{3\pi h^3}{25}$

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$\frac{dV}{dh} = \frac{9\pi h^2}{25}$

$1.6 \frac{m^3}{min} = \frac{9\pi h^2}{25} \frac{m^3}{m} \cdot \frac{dh}{dt}$

$1.6 \cdot \frac{25}{9\pi h^2} = \frac{dh}{dt} \rightarrow \frac{dh}{dt} = 1.6 \cdot \frac{25}{9\pi (2)^2} \doteq \boxed{0.35\, m/min}$

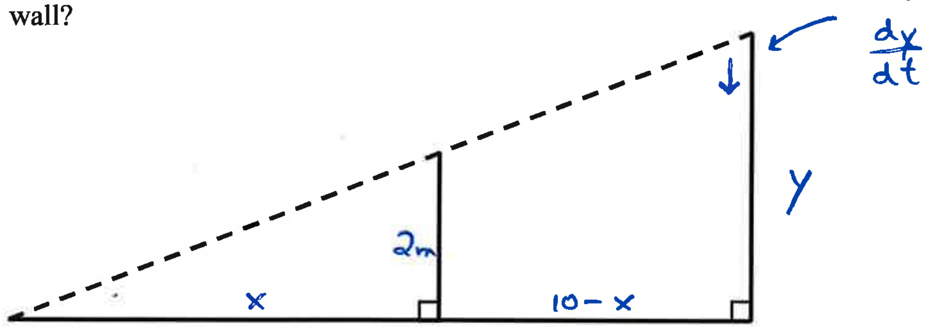
Tips for Solving Related Rates Problems

- Read and reread the problem until you are sure you understand it.
- Drawing a diagram may be helpful.
- Assign variables to all quantities in the function you are using.
- Differentiate both sides of the equation writing the required rate in terms of derivatives.
- Solve for the rate you are interested in.
- Substitute and all known values and calculate the rate.

Be sure to differentiate first, then solve, ***then*** substitute. It is much easier to do algebraic manipulation symbolically as opposed to moving numbers around.

Ex. 3

A spotlight on the ground shines on a wall 10 m away. A man 2 m tall walks from the spotlight toward the wall at a speed of 1.2 m/s. How fast is his shadow on the wall decreasing when he is 3 m from the wall?



$10 - x = 3$

so $x = 7$

$\frac{dx}{dt} = 1.2\ m/s$

again use similar triangles

$\frac{y}{10} = \frac{2}{x}$

$y = \frac{20}{x}$

$\frac{dy}{dt} = \frac{dx}{dt} \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{20}{x^2}$

$\frac{dy}{dt} = 1.2\left(\frac{-20}{x^2}\right)$ at $x = 7$

$\frac{dy}{dt} = 1.2\left(\frac{-20}{49}\right) = -\frac{24}{49}$

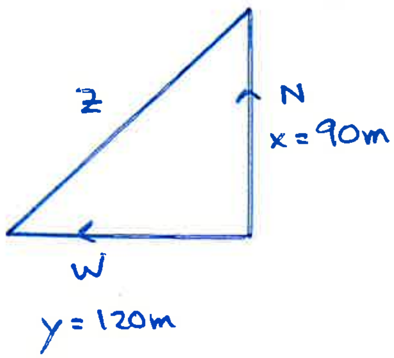
so the shadow is decreasing at a rate of $\frac{24}{49}$ m/s

Ex. 4

A man starts walking north at a speed of 1.5 m/s and a woman starts at the same point *P* at the same time walking west at a speed of 2 m/s. At what rate is the distance between the man and the woman increasing one minute later?

$1.5\text{m/s} \cdot 60 \text{ secs} = 90\text{m}$

$2\text{m/s} \cdot 60 \text{ secs} = 120\text{m}$



$\frac{dx}{dt} = 1.5\text{m/s}$ need $\frac{dz}{dt}$

$\frac{dy}{dt} = 2\text{ m/s}$

implicit differentiation

$z^2 = x^2 + y^2$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$z^2 = 90^2 + 120^2$

$z^2 = 8100 + 14400$

$z^2 = 22500$

$z = 150$

$2(150)\frac{dz}{dt} = 2(90)(1.5) + 2(120)(2)$

$300\frac{dz}{dt} = 270 + 480$

$300\frac{dz}{dt} = 750$

$\frac{dz}{dt} = \frac{750}{300} = \frac{5}{2} = 2.5$

$\boxed{\text{Rate between them is increasing at } 2.5\text{ m/s}}$

Homework Assignment

- Section 3.5: #1 – 7, 9, 11, 12, 14, 15

Section 3.6 – Newtons Method has been Omitted from this Booklet

Chapter Review

- Section 3.7: #1 – 7, 9, 10