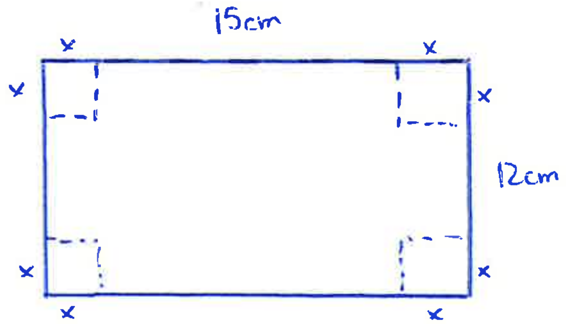
Section 3.5 – Practice Problems

1. An open top rectangular box is constructed by cutting a square length x from each corner of a $12cm\ by\ 15cm$ rectangle, and then folding up the sides with $x \geq 2cm$. What size square must be cut to have a volume of $162cm^3$?



$V = lwh$

$l = 15 - 2x$

$w = 12 - 2x$

$h = x$

$162 = (15-2x)(12-2x)x$

$0 = (4x^2 - 54x^2 + 180)x - 162$

$0 = 4x^3 - 54x^2 + 180x - 162$

$0 = 2(2x^3 - 27x^2 + 90x - 81)$

Rational Root: $\dfrac{\pm 1\ \pm 3\ \pm 9\ \pm 27\ \pm 81}{\pm 1\ \pm 2}$

Test roots

$P(3) = 0$ so $x - 3$ is a root

3	2	−27	90	−81
		6	−63	81
	2	−21	27	0

$(x-3)(2x^2 - 21x + 27) \rightarrow (x-3)(2x-3)(x+9)$

$x = 3, \frac{3}{2}, -9$ Reject $\frac{3}{2}, -9$

$x \geq 2$

So $\boxed{x = 3}$

2. What length must be cut if the volume of the box in question 1 is $150cm^3$.

$150 = (15-2x)(12-2x)x$

$0 = 4x^3 - 54x^2 + 180x - 150$

$0 = 2(2x^3 - 27x^2 + 90x - 75)$

Rational Root

$\dfrac{\pm 1\ \pm 3\ \pm 5\ \pm 15\ \pm 25\ \pm 75}{\pm 1\ \pm 2}$

Does not factor easily roots are tough to find. Use Desmos.

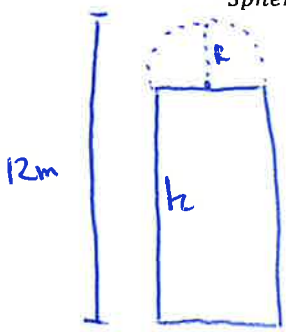
$x = 1.275 \leftarrow x \geq 2$

$x = 3.293 \leftarrow$ this is the answer

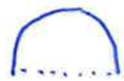
$x = 8.932 \leftarrow$ too big

$\boxed{x = 3.293}$

3. A silo is shaped like a cylinder topped by a hemisphere. The overall height of the silo is $12m$. Find the radius if the volume is $360\pi m^3$. $V_{Cylinder} = \pi r^2 h$ and $V_{Sphere} = \frac{4}{3}\pi r^3$



Hemisphere Top



$V_{cyl} = \pi r^2 h$

$V_{sphere} = \frac{4}{3}\pi r^3$

$h + r = 12 \rightarrow h = 12 - r$

Total Volume $= V_{CYL} + V_{HEMISPHERE}$

$360\pi = \pi r^2 h + \left(\frac{4}{3}\pi r^3\right)\frac{1}{2}$

$360\pi = \pi r^2(12 - r) + \frac{4}{6}\pi r^3$

$360\pi = \pi(12r^2 - r^3) + \frac{2}{3}\pi r^3$

cancel out π

$360 = 12r^2 - r^3 + \frac{2}{3}r^3$

$0 = -\frac{1}{3}r^3 + 12r^2 - 360$ Divide out $-\frac{1}{3}$

$0 = r^3 - 36r^2 + 1080$

$\pm 1 \ \pm 2 \ \pm 3 \ \pm 5 \ \pm 6 \ldots$ Rational Root Theorem

$P(6) = 0$

$$\begin{array}{c|cccc} 6 & 1 & -36 & 0 & 1080 \\ & & 6 & -180 & -1080 \\ \hline & 1 & -30 & -180 & 0 \end{array}$$

$(r - 6)(r^2 - 30r - 180)$ by QE

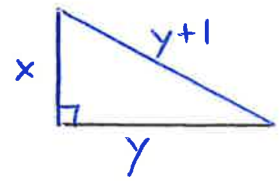
$r = 15 \pm 9\sqrt{5}$

$\boxed{r = 6}$ Reject both too big and negative

4. A right triangle has a hypotenuse $1cm$ longer than one of the sides. Find the length of the sides if the area of the triangle is $6cm^2$.

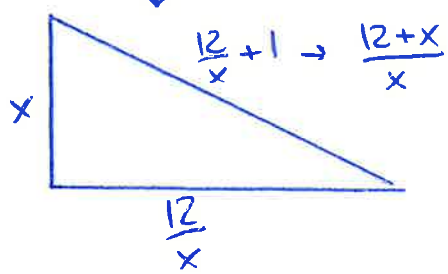
Pythagorean Theorem $= a^2 + b^2 = c^2$

Area of a Triangle $= \frac{1}{2}bh$



$\frac{1}{2}bh = 6 \rightarrow xy = 12$

$y = \frac{12}{x}$



$a^2 + b^2 = c^2$

$x^2 + \left(\frac{12}{x}\right)^2 = \left(\frac{12 + x}{x}\right)^2$

$x^2 + \frac{144}{x^2} = \frac{144 + 24x + x^2}{x^2}$

$\frac{x^4 + 144}{x^2} = \frac{144 + 24x + x^2}{x^2} \rightarrow x^4 - x^2 - 24x = 0$

$x(x^3 - x - 24) = 0$

$$\begin{array}{c|cccc} 3 & 1 & 0 & -1 & -24 \\ & & 3 & 9 & 24 \\ \hline & 1 & 3 & 8 & 0 \end{array}$$

By Rational Root Remainder

$P(3) = 0$

$(x - 3)(x^2 + 3x + 8)$

Qudratic Eqn

$\frac{-3 \pm \sqrt{3^2 - 4(1)(8)}}{2(1)}$

$\frac{-3 \pm \sqrt{-23}}{2}$

NO SOLUTION

So;

$\boxed{x = 3}$

$y = \frac{12}{x}$

$y = \frac{12}{3}$

$\boxed{y = 4}$

$\boxed{y + 1 = 5}$

5. A box is $1m$ *by* $2m$ *by* $3m$. If each side is increased by the same amount, how much must you increase these sides to make the volume 10 *times* larger.

$V = 1\cdot 2\cdot 3$

$V = 6$

$10V = (1+x)\underbrace{(2+x)(3+x)}_{\text{FOIL}}$

$10(6) =$

$10(6) = (x^2+3x+2)(3+x)$

$60 = x^3+3x^2+3x^2+9x+2x+6$

$0 = x^3+6x^2+11x-54$

Rational Root: $\dfrac{\pm 1 \pm 2 \pm 3 \ldots}{\pm 1}$

$P(2) = 0$

2	1	6	11	−54
		2	16	54
	1	8	27	0

$(x-2)(x^2+8x+27)$

QE

$\dfrac{-8 \pm \sqrt{64-4(1)(27)}}{2}$

$\dfrac{-8 \pm \sqrt{-44}}{2} = \varnothing$

$x = 2$

increase sides by 2

6. A box measures $(1 \cdot 1 \cdot 2)m$. Each side is increased the same amount. How much is this increase if the volume is increased by six times the original volume.

$V = 1\cdot 1\cdot 2$

$V = 2$

$6V = (1+x)(1+x)(2+x)$

$12 = (x^2+2x+1)(2+x)$

$12 = 2x^2+4x+2+x^3+2x^2+x$

$0 = x^3+4x^2++5x+2-12$

$0 = x^3+4x^2+5x-10$

Rational Root: $\pm 1 \pm 2 \pm 5 \pm 10$

$P(1) = 1^3+4(1)^2+5(1)-10$

$= 10-10$

$= 0$

1	1	4	5	−10
		1	5	10
	1	5	10	0

$(x-1)(x^2+5x+10)$

QE

$\dfrac{-5 \pm \sqrt{25-4(1)(10)}}{2(1)}$

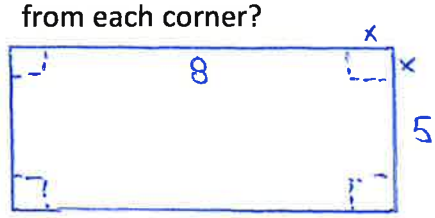
$\dfrac{-5 \pm \sqrt{-15}}{2} = \varnothing$

$\boxed{x = 1}$

Sides become

$\boxed{2\cdot 2\cdot 3}$

7. An open top box is made from a piece of cardboard measuring $5in \ x\ 8in$. Cutting out squares from each corner and folding the edges up makes a box with a volume of $14in^3$. How large a square must be cut from each corner?



$V = l \cdot w \cdot h \qquad 14 = (8-2x)(5-2x)x$

$14 = (40 - 26x + 4x^2)x$

$0 = 40x - 26x^2 + 4x^3 - 14$

$4x^3 - 26x^2 + 40x - 14 = 0$

$2(2x^3 - 13x^2 + 20x - 7) = 0$

RR Theorem $\dfrac{\pm 1 \ \pm 7}{\pm 1 \ \pm 2}$

$P(1) = 2(1)^3 - 13(1)^2 + 20(1) - 7$
$= 2 - 13 + 20 - 7$
$=$ NO GOOD

$P(-1) =$ NO
$P(7) =$ NO
$P(-7) =$ NO

$P(\frac{1}{2}) = 0$

$$\begin{array}{c|cccc} \frac{1}{2} & 2 & -13 & 20 & -7 \\ & & 1 & -6 & 7 \\ \hline & 2 & -12 & 14 & 0 \end{array}$$

$(x - \frac{1}{2})(2x^2 - 12x + 14)$

Factor of 2 from into

$(2x-1)(x^2 - 6x + 7)$ QE $\rightarrow \dfrac{6 \pm \sqrt{36 - 4(1)(7)}}{2}$

$\dfrac{6 \pm \sqrt{8}}{2}$

reject too big $\rightarrow$ $x = 4.4$

$x = 1.6$

$\boxed{x = \frac{1}{2} \quad x = 1.6}$

8. The production of $x\ units$ produces revenue. $R(x) = 100x - x^2$ and costs of $C(x) = \dfrac{1}{3}x^3 - 6x^2 + 89x + 100$. At what point does the company make a profit? (Profit begins when $Revenue = Cost$)

Profit starts when $R(x) - C(x) \geq 0$

$100x - x^2 - (\frac{1}{3}x^3 - 6x^2 + 89x + 100) \geq 0$

$100x - x^2 - \frac{1}{3}x^3 + 6x^2 - 89x - 100 \geq 0$

$-\frac{1}{3}x^3 + 5x^2 + 11x - 100 \geq 0$ (multiply by -3)

$\downarrow$

$x^3 - 15x^2 - 33x + 300 \geq 0$

Tough to Factor so use DESMOS and interpret.

Roots: $x = -4.812$ $\leftarrow$ reject negative value

$x = 3.924$

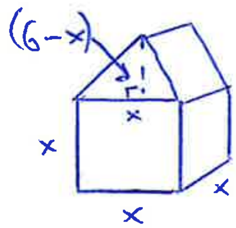
$x = 15.889$

$P(x) \geq 0$ between

$\boxed{3.924 \rightarrow 15.889 \text{ units}}$

9. A shed is constructed in the shape of a cube with a triangular prism forming the roof. The total height of the shed is $6m$, with a volume of $80m^3$. Find the length of the sides of the shed. $Volume_{Cube} = x^3$

$$Volume_{Triangular\ Prism} = \frac{1}{2}lwh$$



Volume cube $= x^3$

Volume TP $= \frac{1}{2}lwh$

$l = x$

$w = x$

$h = (6-x)$

Total Volume:

$x^3 + x \cdot x(6-x) \rightarrow x^3 + (x^2(6-x))\frac{1}{2}$

$= x^3 + \frac{1}{2}(6x^2 - x^3)$

$= x^3 + 3x^2 - \frac{x^3}{2}$

$80 = x^3 + 3x^2 - \frac{x^3}{2}$

multiply by 2

$\downarrow$

$160 = 2x^3 + 6x^2 - x^3 \rightarrow x^3 + 6x^2 - 160 = 0$

RR Theorem.

$P(4) = 0$

$$\begin{array}{r|rrrr} 4 & 1 & 6 & 0 & -160 \\ & & 4 & 40 & 160 \\ \hline & 1 & 10 & 40 & 0 \end{array}$$

$(x-4)(x^2+10x+40) \leftarrow$ QE

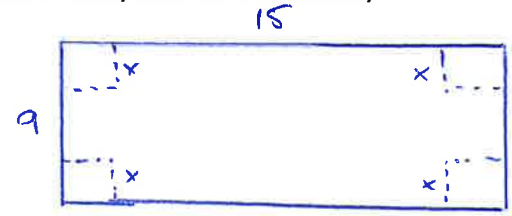
$\boxed{x=4}$

$\frac{-10 \pm \sqrt{100-160}}{2}$

$\downarrow$

$\varnothing$

10. An open box is made from a piece of cardboard $9in\ x\ 15in$, by cutting equal square corners and turning up the sides. Find the maximum volume of the box. (Think Max/Min for Parabolas)



$V = x(15-2x)(9-2x)$

$V = x(135 - 30x - 18x + 4x^2)$

$V = 135x - 30x^2 - 18x^2 + 4x^3$

$V = 4x^3 - 48x^2 + 135x$

Use Desmos now.

Look for Max values for y-values at a given x.

Max occurs at $x = 1.821$

$\boxed{y = 110.8}$

$\uparrow$

That's the max volume.

11. A box has a square base; the perimeter of the base plus the height is $120cm$. What length of the base yields a volume of $13\,500cm^3$?

Box is $x \cdot x$ so Perimeter of Base: $4x$

$h = ?$

P: $4x + h = 120 \rightarrow h = 120 - 4x$

V: $x^2h = 13500$

$V = x^2(120-4x) = 13500$

$120x^2 - 4x^3 - 13500 = 0$

$-4x^3 + 120x^2 - 13500 = 0$ (Divide by -4)

$x^3 - 30x^2 + 3375 = 0$

RR Theorem

$P(15) = 0$

$$\begin{array}{r|rrrr} 15 & 1 & -30 & +0 & +3375 \\ & & 15 & -225 & -3375 \\ \hline & 1 & -15 & -225 & 0 \end{array}$$

$(x-15)(x^2-15x-225)$ QE $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\frac{15 \pm \sqrt{15^2-4(1)(-225)}}{2}$

$\frac{15 \pm \sqrt{1125}}{2}$

$\frac{15 \pm 15\sqrt{5}}{2}$

$\boxed{\begin{array}{l} x = 15 \\ x = 24.27 \end{array}}$

$x = 24.27$

$x = -9.27$ ← Reject negative

12. Calculate the maximum volume of the box in the previous question. To achieve this maximum, what are the dimensions of the box?

Plug $V = x^2(120-4x) = \boxed{120x^2 - 4x^3}$

into Desmos

and find maximum (local)

y-values at particular x-values

at

$x = 20$

$y = 16000\,cm^3$

$4x + h = 120$

$4(20) + h = 120$

$h = 120 - 80$

$h = 40$

so at base $x = 20\,cm$

height $y = 40\,cm$

V is maxed at $16000\,cm^3$

See Website for Detailed Answer Key

Extra Work Space