Section 3.4 – Remainder and Factor Theorems

The Remainder Theorem

- If we consider the divided form of a polynomial:

$$\boldsymbol{f(x) = (x-a)q(x) + r,} \qquad \textbf{\textit{where r is a constant or 0}}$$

- If we evaluate $f(x)$ at $x = a$ then:

$$\boldsymbol{f(a) = (a-a)q(x) + r}$$

$$\boldsymbol{f(a) = r}$$

- What this means is that the **value of a polynomial at $\boldsymbol{x = a}$** is the **same as the remainder** when we **divide by $\boldsymbol{(x-a)}$**. This has **big connotations for a remainder of zero!**

Remainder Theorem

If the polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

Example 1: Without dividing, what is the remainder when $P(x) = 2x^4 - 3x^3 + 2x - 3$ is divided by $x - 2$?

Solution 1:

By Remainder Theorem	By Synthetic Division
$P(x) = 2x^4 - 3x^3 + 2x - 3$ $P(2) = 2(2)^4 - 3(2)^3 + 2(2) - 3$ $P(2) = 9$	$\begin{array}{c\|ccccc} 2 & 2 & -3 & 0 & 2 & -3 \\ & & 4 & 2 & 4 & 12 \\ \hline & 2 & 1 & 2 & 6 & 9 \end{array}$ Remainder: 9

Example 2: For what value of k will the remainder be 5 when $P(x) = x^3 - 2x^2 + x + k$ is divided by $x - 2$?

Solution 2:

<table>
<tr><th>By Remainder Theorem</th><th>By Synthetic Division</th></tr>
<tr><td>

$P(x) = x^3 - 2x^2 + x + k$

$P(2) = (2)^3 - 2(2)^2 + (2) + k = 5$

$8 - 8 + 2 + k = 5$

$k = 3$

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$$\begin{array}{r|rrrr} 2 & 1 & -2 & 1 & k \\ & & 2 & 0 & 2 \\ \hline & 1 & 0 & 1 & k+2 \end{array}$$

Remainder ($k+2$)

$k + 2 = 5 \quad \rightarrow \quad k = 3$

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Example 3: Find the remainder when $x^{17} - 2x^{12} + 7$ is divided by $x + 1$

Solution 3: By Synthetic Division, this would be a nightmare, so the Remainder Theorem is extremely helpful!

Divided by $x + 1$ means that $x = -1$, so calculate $P(-1)$ to get the remainder

$$P(x) = x^{17} - 2x^{12} + 7$$

$$P(-1) = (-1)^{17} - 2(-1)^{12} + 7$$

$$P(-1) = -1 - 2 + 7 = 4$$

The Remainder is 4.

Next, we will see how having a **Remainder of Zero** helps us **determine a Factor**

The Factor Theorem

- This **follows directly** from the **Remainder Theorem** and was alluded to.
- If the Polynomial $P(x)$ is $\boldsymbol{divded\ by}$ $(\boldsymbol{x-a})$ and the **remainder is zero**, then $(\boldsymbol{x-a})$ **divides the Polynomial** and **is a Factor** of the Polynomial

<u>Example 4:</u> Given $P(x) = 3x^4 + 4x^3 - 3x^2 - 3x - 10$. Is $(x+2)$ a factor of $P(x)$?

<u>Solution 4:</u> Using the Remainder Theorem, it should follow that if $(x+2)$ is a factor, then should give us: $P(-2) = 0$

$$P(-2) = 3(-2)^4 + 4(-2)^3 - 3(-2)^2 - 3(-2) - 10$$

$$P(-2) = 3(16) + 4(-8) - 3(4) - 3(-2) - 10$$

$$\boldsymbol{P(-2) = 48 - 32 - 12 + 6 - 10 = 0}$$

So that means that **yes,** $(\boldsymbol{x+2})$ **is a factor** and we can use Synthetic Division to factor it out.

-2	3	4	-3	-3	-10
		-6	4	-2	10
	3	-2	1	-5	0

We end up with:

$$(x+2)(3x^3 - 2x^2 + x - 5)$$

In the next Section, we will see how we can estimate potential roots so we could continue this process

<u>Example 5:</u> Given $P(x) = 3x^4 + 4x^3 - 3x^2 - 3x - 10$. Is $(x-1)$ a factor of $P(x)$?

<u>Solution 5:</u> Using the Remainder Theorem, it should follow that if $(x-1)$ is a factor, then should give us: $P(1) = 0$

$$P(1) = 3(1)^4 + 4(1)^3 - 3(1)^2 - 3(1) - 10$$

$$\boldsymbol{P(1) = 3 + 4 - 3 - 3 - 10 = -9}$$

So that means that **no,** $(\boldsymbol{x-1})$ **is not a factor** and we can use Synthetic Division to check.

1	3	4	-3	-3	-10
		3	7	4	1
	3	7	4	1	-9

We end up with something we cannot factor, with the given parameters.

- From the above examples, we can state that:

Factor Theorem

If P is a continuous Polynomial Function. Then $P(x)$ has a factor of $(x - a)$ if, and only if, $P(a) = 0$. The if and only if portion of the definition implies two separate but necessary parts of the Theorem.

1. $If\ P(a) = 0$ then $(x - a)$ is a factor of $P(x)$
2. $If\ \ (x - a)$ is a factor of $P(x)$, then $P(a) = 0$

Example 6: Given $P(x) = x^3 - 2x^2 + 3x - 6$. Is $(x - 2)$ a factor of $P(x)$?

Solution 6: Using the Remainder Theorem, it should follow that $P(2) = 0$ if $(x - 2)$is a factor

$$P(2) = (2)^3 - 2(2)^2 + 3(2) - 6$$

$$\boldsymbol{P(2) = 8 - 8 + 6 - 6 = 0}$$

So that means that **yes, $(x - 2)$ is a factor** and we can use Synthetic Division to check.

2	1	−2	3	−6
		2	0	6
	1	0	3	0

Since **the Remainder is Zero**, then $(x - 2)$ is a factor:

$$(x - 2)(x^2 + 3)$$

Example 7: Given $P(x) = 2x^4 - 3x^2 + x - 1$. Is $(x + 1)$ a factor of $P(x)$?

Solution 7: Using the Remainder Theorem, it should follow that $P(-1) = 0$ if $(x + 1)$is a factor

$$P(-1) = 2(-1)^4 - 3(-1)^2 + (-1) - 1$$

$$\boldsymbol{P(-1) = 2 - 3 - 1 - 1 = -3}$$

So that means that **no, $(x + 1)$ is not a factor** and we can use Synthetic Division to check.

−1	2	0	−3	1	−1
		−2	2	1	−2
	2	−2	−1	2	−3

We end up with something we cannot factor, with the given parameters.

The Rational Root Theorem

- This is the last part to our work
- It allows us to estimate potential roots, then we can use the Remainder and Factor Theorem to fully factor our higher degree polynomials

Consider a $Polynomial\ P(x) = 3x^2 + 2x - 8$ if we recall Binomial Multiplication, we would be looking for the first two terms to multiply to 3 and the last two terms multiply to -8

That means we have the following possibilities:

Possible Factors $\mathbf{3x^2 + \cdots - 8}$	**Corresponding Zeros**
$(3x \pm 1)(x \pm 8)$	$\pm\frac{1}{3}$ *and* ± 8
$(3x \pm 2)(x \pm 4)$	$\pm\frac{2}{3}$ *and* ± 4
$(3x \pm 4)(x \pm 2)$	$\pm\frac{4}{3}$ *and* ± 2
$(3x \pm 8)(x \pm 1)$	$\pm\frac{8}{3}$ *and* ± 1

- What this demonstrates is that the possible roots are related to the possible factors of the first and last terms.

$$\frac{possible\ factors\ of\ last\ term}{possible\ factors\ of\ first\ term} = \frac{\pm1, \pm2, \pm4, \pm8}{\pm1, \pm3}$$

$$\left\{\pm1, \pm2, \pm4, \pm8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}\right\}$$

Once you have this list, pick values and start trying them using the Remainder Theorem!

You will get good at picking strategically, it should be apparent that ±1 will not work, so try $x = -2$

$P(-2) = 3(-2)^2 + 2(-2) - 8$

$P(-2) = 12 - 4 - 8 = 0$ ← Once you have a Root, **use Synthetic Division to start Factoring**!

Example 8: Find the zeros (roots) of $f(x) = x^3 - 9x^2 + 20x - 12$

Solution 8: By the Rational Root Theorem, potential roots are:

Twelve possible roots!

$$\frac{factors\ of\ 12}{factors\ of\ 1} = \frac{\pm1, \pm2, \pm3, \pm4, \pm6, \pm12}{\pm1} = \pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \pm\mathbf{4}, \pm\mathbf{6}, \pm\mathbf{12}$$

- Start with smaller numbers, 1, 2 if available.

$$\boldsymbol{f(1) = (1)^3 - 9(1)^2 + 20(1) - 12 = 1 - 9 + 20 - 12 = 0}$$

We have a **remainder of zero when $\boldsymbol{x = 1}$** so $(\boldsymbol{x - 1})$ **is a root**, use Synthetic Division to factor.

1	1	−9	20	−12
		1	−8	12
	1	−8	12	0

We have: $(x-1)(x^2 - 8x + 12)$

Which can be factored easily to:

$$(x-1)(x-2)(x-6)$$

Therefore, of the 12 possible roots, we get: $x = 1, 2, 6$

Example 9: Find the zeros (roots) of $f(x) = 4x^3 + 12x^2 + 5x - 6$

Solution 9: By the Rational Root Theorem, potential roots are:

16 possible roots!

$$\frac{factors\ of\ 6}{factors\ of\ 4} = \frac{\pm1, \pm2, \pm3, \pm6}{\pm1, \pm2, \pm4} = \pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \pm\mathbf{6}, \pm\frac{\mathbf{1}}{\mathbf{2}}, \pm\frac{\mathbf{1}}{\mathbf{4}}, \pm\frac{\mathbf{3}}{\mathbf{2}}, \pm\frac{\mathbf{3}}{\mathbf{4}}$$

- Start with smaller numbers, 1, 2 if available. In this case $x = 1\ and\ x = -1$ do not work.

$$\boldsymbol{f(-2) = 4(-2)^3 + 12(-2)^2 + 5(-2) - 6 = -32 + 48 - 10 - 6 = 0}$$

We have a **remainder of zero when $\boldsymbol{x = -2}$** so $(\boldsymbol{x + 2})$ **is a root**, use Synthetic Division to factor.

−2	4	12	5	−6
		−8	−8	6
	4	4	−3	0

We have: $(x+2)(4x^2 + 4x - 3)$

Which can be factored by grouping to get:

$$(x+2)(2x-1)(2x+3)$$

Therefore, of the 16 possible roots, we get: $x = -2, \frac{1}{2}, -\frac{3}{2}$

It is always easier to try integer roots before attempting the fractions, they will factor out easier in the end!

Section 3.4 – Practice Problems

1. Find $P(k)$.

a) $P(x) = x^4 + 3x^3 - 7x + 2; k = -2$

b) $P(x) = -2x^4 - 3x^2 - 2; k = \sqrt{2}$

c) $P(x) = -2x^2 + 4x + 3; k = 1 + \sqrt{2}$

d) $P(x) = x^5 - 5a^4x + 4a^5; k = a$

2. Use the Remainder Theorem to solve for $k\ and\ m$.

a) When $x^3 + kx + 1$ is divided by $x - 2$, the remainder is -3

b) When $x^3 - x^2 + kx - 8$ is divided by $x - 4$, the remainder is 0

c) When $2x^4 + kx^2 - 3x + 5$ is divided by $x - 2$, the remainder is 3

d) When $x^3 + kx + 6$ is divided by $x + 2$, the remainder is 4

e) When $x^3 + kx^2 - 2x - 7$ is divided by $x + 1$, the remainder is 5. What is the remainder when it is divided by $x - 1$?

f) When $kx^3 + mx^2 + x - 2$ is divided by $x - 1$, the remainder is 6. When the it is divided by $x + 2$, the remainder is 12.

g) When $x^4 + kx^3 - mx + 15$ has no remainder when divided by $x - 1$ and $x + 3$.

h) When $P(x) = 3x^4 + kx^2 + 7$ is divided by $x - 1$, the remainder is the same as when $f(x) = x^4 + kx - 4$ is divided by $x - 2$

3. Use the Remainder Theorem to solve the following:

a) If a Polynomial equation $P(x)$ is divded by $x - a$, what is the value of its remainder?

b) Given $P(x) = x^3 - rx^2 + 3x + r^2$, find all possible values of r so that $P(3) = 18$

c) When the polynomial $x^n + x - 8$ is divided by $x - 2$, the remainder is 10. What is the value of n.

d) When $x^2 + 5x - 2$ is divided by $x + a$, the remainder is 8. Find all possible values of a

e) When the polynomial $P(x) = kx^{50} + 2x^{30} + 4x + 7$ is divided by $x + 1$, the remainder is 23. What is the value of k.

f) Solve for $k\ and\ m$ if $P(x) = 2x^3 + 3x^2 + kx + m$ and $P(1) = 8$ and $P(-2) = -13$.

4. Find the missing factors by using Synthetic Division

a) $2x^3 - 7x^2 + 2x + 3 = (x - 1)(\qquad)(\qquad)$

b) $x^3 - 3x^2 - 10x + 24 = (x - 2)(\qquad)(\qquad)$

c) $x^4 + x^3 - 9x^2 - 9x = x(x + 1)(\qquad)(\qquad)$

d) $2x^4 - 7x^3 + 9x^2 - 5x + 1 = (x-1)^3(\qquad)(\qquad)$

e) $2x^4 + 5x^3 - 11x^2 - 20x + 12 = (x^2 - 4)(\qquad)(\qquad)$

f) $x^5 - 8x^4 + 25x^3 - 38x^2 + 28x - 8 = (x^2 - 3x + 2)(\qquad)(\qquad)$

5. Solve the following using the Factor Theorem and Rational Root Theorem where necessary

   a) If 2 is a root of the equation $3x^3 + x^2 - 20x + 12 = 0$, determine the other roots

b) What are all values of k for which one-half is a zero of:

$$P(x) = -4x^3 + 2x^2 - 2kx + k^3$$

c) If $x = c$ is a root of the polynomial equation $P(x) = 0$, then what must be a factor of $P(x)$.

d) If $x - a$ is a factor of $2x^3 - ax^2 + (1 - a^2)x + 5$, what is a

e) Determine k so that $x+1$ is a factor of: $2x^4+(k+1)x^2-6kx+11$

f) A polynomial has factors: $x^2-4, x^2-2x, and\ x^2+x-2$. What is the lowest possibl degree of the polynomial?

g) For what number k is k a zero of: $f(x) = 2x^3 - kx^2 + (3 - k^2)x - 6$?

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h) Find the complete factored form of the following:

$$P(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$$

If $P(-2) = P(-1) = P(1) = 0$

i) Factor completely:

$$P(x) = x^5 - 3x^4 + 8x^2 - 9x + 3$$

If $x - 1$ is a factor

j) Determine values for $a \; and \; b$ such that $x - 1$ is a factor of both:

$$x^3 + x^2 + ax + b \qquad \text{and} \qquad x^3 - x^2 - ax + b$$

See Website for Detailed Answer Key

Extra Work Space