3.4 Rates of Change in the Social Sciences (Economics)

The **cost function** $C(x)$ is the amount of money required to produce x units of a certain commodity. If the number of items produced is increased from x_1 to x_2, the *additional cost* is $\Delta C = C(x_2) - C(x_1)$ and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

Taking the limit as $\Delta x \to 0$ we obtain the instantaneous rate of change of cost with respect to the number of items produced, which economists call the **marginal cost**. In reality x is an integer so it does not make sense to allow Δx to approach 0 unless we approximate the cost function as a smooth function like we did in the previous section with the population function.

Derivative of cost function →

$$\text{Marginal Cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

In this way, the marginal cost is the derivative of the cost function. To interpret this rate of change better let us examine it using the definition of derivative.

$$C'(n) = \lim_{h \to 0} \frac{C(n+h) - C(n)}{h}$$

Letting $h = 1$ and assuming n is a large number, the above is approximately equal to

$$C'(n) \doteq C(n+1) - C(n)$$

Which means that the marginal cost of producing n units is approximately equal to the cost of producing one more unit, the $(n+1)$st unit.

Cost functions are often represented by a polynomial where a represents the fixed cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labour, and so on. The cost of raw materials may be proportional to x, but labour costs might depend on higher powers of x because of overtime costs and inefficiencies in large-scale operations.

Ex. 1

Quinton Mills is a large producer of flour. Management estimates that the cost (in dollars) of production x 5 kg bags of flour is

$$C(x) = 140\,000 + 0.43x + 0.000\,001x^2$$

(a) Find the marginal cost at a production level of $x = 1000$ bags.
(b) Find the actual cost of producing the 1001st bag.

a) $C'(x) = 0.43 + 2(0.000\,001x)$

$= 0.43 + 0.000\,002x$

$C'(1000) = 0.43 + 0.000\,002(1000)$

$= \$0.432/\text{bag}$

b) $C(1001) - C(1000)$

$140\,000 + 0.43(1001) + 0.000\,001(1001)^2 - [140\,000 + 0.43(1000) + 0.000\,001(1000)^2]$

$140\,431.432 - 140\,431 = \$0.432$

price per unit

Economists are also interested not only in costs but revenue and profit. Let $p(x)$ be the price per unit that a company can charge if it sells x units. Then p is called the **demand function** (or **price function**). If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$\boxed{R(x) = xp(x)}$$

number of units

And R is the **revenue function**. The derivative R, R' is called the **marginal revenue function** and is the rate of change of revenue with respect to the number of units sold.

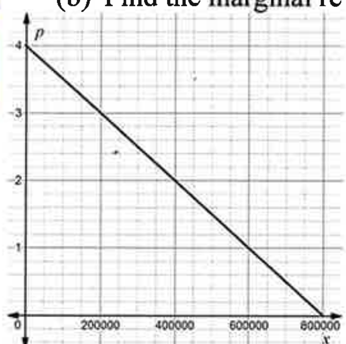
Ex. 2

Howard's Hamburgers has taken a market survey and has found that the yearly demand for their hamburgers is given by

$$p(x) = \frac{800\,000 - x}{200\,000} \qquad (p \text{ in dollars})$$

(a) What is the demand for hamburgers corresponding to prices of \$0.00, \$0.50, \$1.00, \$1.50, \$2.00, \$2.50, \$3.00, \$3.50, \$4.00?

(b) Find the marginal revenue when $x = 300\,000$



more burgers are sold as the price decreases

x	8×10^5	7×10^5	6×10^5	5×10^5	4×10^5	3×10^5	2×10^5	1×10^5	0
p	\$0	\$0.50	\$1.00	\$1.50	\$2.00	\$2.50	\$3.00	\$3.50	\$4.00

$R(x) = xp(x)$

$= x\left(\dfrac{800\,000 - x}{200\,000}\right)$

$\dfrac{800\,000x - x^2}{200\,000} = \dfrac{1}{200\,000}(800\,000x - x^2)$

$R'(x) = \dfrac{1}{200\,000}(800\,000 - 2x)$

$R'(300\,000) = \dfrac{1}{200\,000}(800\,000 - 600\,000) = \boxed{\$1/\text{hamburger}}$

If x units are sold of a commodity, the total profit is obtained by subtracting the cost from the revenue:

$$\boxed{P(x) = R(x) - C(x)}$$

Profit Function

Here P is called the **profit function**. The **marginal profit** function is P', the derivative of the profit function.

Ex. 3

Howard's Hamburgers estimates that the cost, in dollars, or making x hamburgers is

$$C(x) = 125\,000 + 0.42x$$

Using the demand function from Example 2, find the profit and the marginal profit when

(a) $x = 300\,000$ (b) $x = 400\,000$

$$C(x) = 125\,000 + 0.42x$$

$$R(x) = \frac{1}{200\,000}(800\,000x - x^2) = 4x - \frac{x^2}{200\,000}$$

$$P(x) = R(x) - C(x) = 4x - \frac{x^2}{200\,000} - (125\,000 + 0.42x)$$

$$\boxed{P(x) = \frac{-x^2}{200\,000} + 3.58x - 125\,000}$$

$$\boxed{P'(x) = \frac{-x}{100\,000} + 3.58}$$

a) $P(300\,000) = -\frac{300\,000^2}{200\,000} + 3.58(300\,000) - 125\,000 = \boxed{\$499\,000}$

$P'(300\,000) = \frac{-300\,000}{100\,000} + 3.58 = \boxed{\$0.58/\text{hamburger}}$

b) $P(400\,000) = \frac{-400\,000^2}{200\,000} + 3.58(400\,000) - 125\,000 = \boxed{\$507\,000}$

$P'(400\,000) = \frac{-400\,000}{100\,000} + 3.58 = \boxed{\$-0.42/\text{hamburger}}$

increases revenue but decreases profit!

Homework Assignment

- Section 3.4: #1 – 6