3.3 Rates of Change in the Natural Sciences

Recall that the derivative of a function can in general be interpreted as the instantaneous rate of change of that function with respect to its independent variable. This concept has applications in other branches of physics than just ***kinematics***, and in biology, and in chemistry.

If y is a quantity that depends on another quantity x, we can write y as a function of x: $y = f(x)$. If x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

And the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The average rate of change of y with respect to x is defined as

$$\frac{\Delta y}{\Delta x}$$

Which is just the slope of the line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

The **instantaneous rate of change** of y with respect to x can be calculated by allowing $\Delta x \to 0$, or equivalently, allowing $x_2 \to x_1$. Written as a limit we have

$$\text{Instantaneous Rate of Change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1)$$

Which is simply the value of the derivative of f evaluated at $x = x_1$.

Ex. 1
A spherical balloon is being inflated. Find the rate of change of the volume with respect to the radius when the radius is 10 cm.

$V = \frac{4\pi r^3}{3}$

$V' = 4\pi r^2$ at $r = 10$

$= 4\pi(10)^2$

$= \boxed{400\pi \ \text{cm}^3/\text{cm}}$

Applications to Physics

The **linear density** (ρ) of a thin rod or wire is defined as the mass per unit length:

$$\rho = \frac{m}{L}$$

In SI units (mass in kilograms and length in metres) the linear density ρ is measured in kg/m. If the mass of the rod is not ***homogeneous***, let $m = f(x)$ be its mass measured from its left end point to any point x as shown in the figure below.

The mass of rod that lies between $x = x_1$ and $x = x_2$ is $\Delta m = f(x_2) - f(x_1)$, so the average linear density of that part of the rod is

$$\text{Average Density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Allowing $\Delta x \to 0$ ($x_2 \to x_1$) we can compute the linear density at point x_1. Symbolically, we can write

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Therefore, the linear density of the rod is the derivative of mass with respect to length.

<u>Ex. 2</u>
Measured from the left-hand side of the rod, the mass in kilograms as a function of length x in metres is given by $m(x) = x^2$.

(a) Find the average density of the part of the rod over the range $2\text{ m} \le x \le 2.3\text{ m}$.
(b) Find the linear density at $x = 2$ m.

a) $\frac{f(x_2) - f(x_1)}{x_2 - x_1} \rightarrow \frac{2.3^2 - 2^2}{2.3 - 2} = \frac{1.29}{0.3} = \boxed{4.3\text{kg}/\text{m}}$

b) $m'(x) = 2x$ at $x = 2$ $\boxed{\rho = 4\text{kg}/\text{m}}$

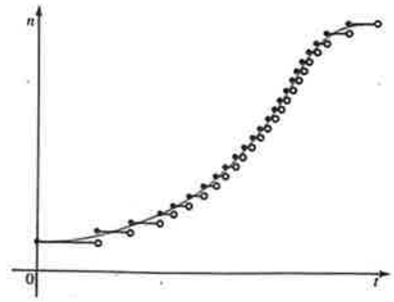
Applications to Biology
We can apply derivatives to understand how large populations grow. If $n = f(t)$ is the number of individuals in a bacteria or animal population at time t, then the change in the population size between times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$ and the average rate of growth can be calculated as follows.

$$\text{Average Rate of Growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

And the **instantaneous rate of growth** is the rate of change of the population size with respect to time.

$$\text{Instantaneous Growth Rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

So, the growth rate is calculated by taking the derivative of the population function. In reality the population is a step function with discontinuities where births and deaths occur as shown in the following graph. However, for large populations a smooth curve can be used to approximate the behaviour of the population.



Ex. 3
The population of a bacteria culture after t hours is given by $n(t) = 500 + 200t + 12t^2$. Find the rate of growth after 5 h.

$n'(t) = 200 + 24t$

at $t = 5$

$n'(5) = 200 + 120$

$n'(5) = \boxed{320 \text{ bacteria/hr}}$

Applications to Chemistry
The *concentration* of a substance A is the number of moles (1 mol = 6.022×10^{23} particles) of a molecule per litre and is denoted $[A]$. When a chemical reaction occurs, the original concentration of substance A decreases as it is used up and converted into a different chemical substance. Because the concentration is a function of time, $[A]$ *decreases* with time t. The average rate of reaction over the time interval $t_1 \leq t \leq t_2$ of a reactant A is given below. Note that the negative sign is used by chemists to make the reaction rate positive.

$$\frac{\Delta[A]}{\Delta t} = -\frac{[A](t_2) - [A](t_1)}{t_2 - t_1}$$

And then the **instantaneous rate of reaction** is the rate of change of the concentration with respect to time.

$$\text{Rate of Reaction} = -\lim_{\Delta t \to 0} \frac{\Delta[A]}{\Delta t} = -\frac{d[A]}{dt}$$

Since the rate of reaction is calculated from the derivative of the concentration function, the slope of the tangent line can be used to calculate the reaction rate (see question #8 from this assignment).

Homework Assignment
- Section 3.3: #1 – 8