Section 3.3 – Dividing Polynomials

- If you enjoyed Long Division in elementary school, you're going to love this!
- Factoring higher degree polynomials will involve find factors, but it is not always as easy as factoring a quadratic. We need tools to help us.
- This section will introduce division (traditional and synthetic) in order to find factors, and the next section will introduce us to two theorems to facilitate the process

First some Vocabulary

$$\begin{array}{r} 3 \\ 2\overline{)\,7} \\ 6 \\ \overline{1} \end{array}$$

2 *is the* ***divisor***

3 *is the* ***quotient***

7 *is the* ***dividend***

1 *is the* ***remainder***

$$7 = 2 \cdot 3 + 1$$

$$\begin{array}{r} q(x) \\ x-a\overline{)\,p(x)} \\ \overline{r} \end{array}$$

$x - a$ *is the* ***divisor***

$q(x)$ *is the* ***quotient***

$p(x)$ *is the* ***dividend***

r *is the* ***remainder***

$$p(x) = (x-a)q(x) + r$$

Long Division

If a polynomial function $f(x) = 6x^3 - 19x^2 + 11x + 6$ has a root at $x = 2$. Then $(x-2)$ is a factor of $f(x)$. So

$$f(x) = (x-2)g(x)$$

We can find the resulting factor $g(x)$ by long division.

$$\begin{array}{r} 6x^2 - 7x - 3 \\ x-2\overline{)\,6x^3 - 19x^2 + 11x + 6} \\ \underline{6x^3 - 12x^2} \qquad\qquad \\ -7x^2 + 11x \qquad \\ \underline{-7x^2 + 14x} \qquad \\ -3x + 6 \\ \underline{-3x + 6} \\ 0 \end{array}$$

Subtract at each step!!!

- Multiply $6x^2(x-2) = 6x^3 - 12x^2$
- Subtract, bring down the next term $11x$
- Multiply $-7x(x-2) = -7x^2 + 14x$
- Subtract, bring down the next term $+6$
- Multiply $-3(x-2) = -3x + 6$
- Subtract and see if there is any remainder

$$f(x) = 6x^3 - 19x^2 + 11x + 6$$

$$f(x) = (x-2)(6x^2 - 7x - 3)$$

$$f(x) = (x-2)(2x-3)(3x+1)$$ Factor the remaining quadratic by Grouping

$$\boldsymbol{x = \frac{3}{2}, -\frac{1}{3}, and\ x = 2}$$

Example 1: Divide $3x^3 - 2x^2 + 1$ *by* $x - 2$

Solution 1: For any missing terms (in descending order) ***insert a 0*** so the subtraction lines up.

$$\begin{array}{r|l} & 3x^2 + 4x + 8 \\ x - 2 & 3x^3 - 2x^2 + 0x + 1 \\ & \underline{3x^3 - 6x^2} \\ & \quad 4x^2 + 0x \\ & \quad \underline{4x^2 - 8x} \\ & \qquad 8x + 1 \\ & \qquad \underline{8x - 16} \\ & \qquad\quad 17 \end{array}$$

Subtract at each step!!!

- Multiply $3x^2(x - 2) = 3x^3 - 6x^2$
- Subtract, bring down the next term $0x$
- Multiply $4x(x - 2) = 4x^2 - 8x$
- Subtract, bring down the next term $+1$
- Multiply $8(x - 2) = 8x - 16$
- Subtract and see if there is any remainder

$f(x) = 3x^3 - 2x^2 + 1$

$3x^3 - 2x^2 + 1 = (x - 2)(3x^2 + 4x + 8) + 17$ or divide both sides by the factor $(x - 2)$ to write it:

$$\frac{3x^3 - 2x^2 + 1}{(x - 2)} = (3x^2 + 4x + 8) + \frac{17}{(x - 2)}$$

Example 2: Divide $3x^4 - 5x^2 + 2x^3 - 1$ *by* $x^2 - 2x + 3$

Solution 2: Write in descending order and ***insert a 0*** for any missing terms to line up subtraction.

$$\begin{array}{r|l} & 3x^2 + 8x + 2 \\ x^2 - 2x + 3 & 3x^4 + 2x^3 - 5x^2 + 0x - 1 \\ & \underline{3x^4 - 6x^3 + 9x^2} \\ & \quad 8x^3 - 14x^2 + 0x \\ & \quad \underline{8x^3 - 16x^2 + 24x} \\ & \qquad 2x^2 - 24x - 1 \\ & \qquad \underline{2x^2 - 4x + 6} \\ & \qquad\quad -20x - 7 \end{array}$$

Subtract at each step!!!

- Multiply $3x^2(x^2 - 2x + 3) = 3x^4 - 6x^3 + 9x^2$
- Subtract, bring down the next term $0x$
- Multiply $8x(x^2 - 2x + 3) = 8x^3 - 16x^2 + 24x$
- Subtract, bring down the next term -1
- Multiply $2(x^2 - 2x + 3) = 2x^2 - 4x + 6$
- Subtract and see if there is any remainder

$f(x) = 3x^4 + 2x^3 - 5x^2 - 1$

$3x^4 + 2x^3 - 5x^2 - 1 = (x^2 - 2x + 3)(3x^2 + 8x + 2) + (-20x - 7)$

Dividend *Divisor* *Quotient* *Remainder*

Synthetic Division

- There is a **shorter and more manageable** way to do division, but many people enjoy Long Division
- It is called **Synthetic Division**, follow the Step-by-Step Guideline in the example below

Example 3: Calculate $3x^3 - 2x^2 + 1 \div (x - 2)$ using Synthetic Division

Solution 3:

Step 1: Write the **dividend in descending order of powers of x**. Then only **copy the coefficients**, remember to **insert 0 for any missing powers of x**.

$$3x^3 - 2x^2 + 1 \quad \rightarrow \quad 3x^3 - 2x^2 + 0x + 1 \quad \rightarrow \quad 3 \quad -2 \quad +0 \quad +1$$

3	-2	0	1	$Row\ 1$

Step 2: Insert **the zero of the divisor to the left**. Since the divisor is $(x - 2)$, the zero is $x = 2$

2	3	-2	0	1	$Row\ 1$
					$Row\ 2$
					$Row\ 3$

Step 3: Bring the first coefficient 3 $\boldsymbol{down\ to\ Row\ 3}$

2	3	-2	0	1	$Row\ 1$
	↓				$Row\ 2$
	3				$Row\ 3$

Step 4: **Multiply** the first entry in Row 3 by the divisor 2 and place the result in Row 2

2	3	-2	0	1	$Row\ 1$
		6			$Row\ 2$
	3				$Row\ 3$

Step 5: **Add** the entry in $Row\ 2$ to the entry above it in $Row\ 1$, and enter the result in $Row\ 3$.

2	3	-2	0	1	$Row\ 1$
		6			$Row\ 2$
	3	4			$Row\ 3$

Step 6: **Repeat this process** until $Row\ 2$ and $Row\ 3$ are filled.

2	3	-2	0	1	$Row\ 1$
		6	8	16	$Row\ 2$
	3	4	8	17	$Row\ 3$
		$Quotient$		$Remainder$	

Step 7: The Quotient is written with each coefficient to a degree lower than the original

$$3x^2 + 4x + 8$$

Quotient: $3x^2 + 4x + 8$

Remainder: 17

We can write it in either way!

$3x^3 - 2x^2 + 1 = (x - 2)(3x^2 + 4x + 8) + 17$

or

$$\frac{3x^3 - 2x^2 + 1}{(x - 2)} = (3x^2 + 4x + 8) + \frac{17}{(x - 2)}$$

$Check$:

$$f(x) = (3x^2 + 4x + 8)(x - 2) + 17$$

$$f(x) = 3x^3 + 4x^2 + 8x - 6x^2 - 8x - 16 + 17$$

$$f(x) = 3x^3 - 2x^2 + 1$$

All Good!

Remember that in **Synthetic Division** we are **ADDING** $Row\ 1\ with\ Row\ 2$

Example 4: Divide $P(x) = 4x^5 - 30x^3 - 50x + 2 \;\; by \;\; x + 3$

Solution 4: **Do not forget the necessary zeros** where we do not have a power, and the root is -3

-3							
	4	0	-30	0	-50	2	$Row\ 1$
		-12	36	-18	54	-12	$Row\ 2$
	4	-12	6	-18	4	-10	$Row\ 3$

So, the result is:

$$\boldsymbol{\frac{4x^5 - 30x^3 - 50x + 2}{x + 3} = 4x^4 - 12x^3 + 6x^2 - 18x + 4 + \frac{-10}{x + 3}}$$

Example 5: Find, using Synthetic Division, k, such that $2x^3 + x^2 - 5x + k$, when divided by $(x + 1)$, has a remainder of -3

Solution 5:

-1					
	2	1	-5	k	$Row\ 1$
					$Row\ 2$
				-3	$Row\ 3$

$\uparrow$ *This is our given remainder*

-1					
	2	1	-5	k	$Row\ 1$
		-2	1	4	$Row\ 2$
	2	-1	-4	-3	$Row\ 3$

So,

$k + 4 = -3 \quad \rightarrow \quad \boldsymbol{k = -7}$

Example 6: Divide $6x^4 - 7x^3 + 4x^2 - 11x + 9$ by $2x - 1$

Solution 6: **IMPORTANT!!** To use Synthetic Division, the coefficient with the x in the divisor has to be 1 so $2x - 1 = 2\left(x - \frac{1}{2}\right)$ so $\boldsymbol{x = \frac{1}{2}}$

But, since we factored out a 2, we have to remember to **divide our answer by that factor!**

$\frac{1}{2}$						
	6	−7	4	−11	9	*Row* 1
		3	−2	1	−5	*Row* 2
	6	−4	2	−10	4	*Row* 3

So, the result is:

$$\frac{6x^4 - 7x^3 + 4x^2 - 11x + 9}{x - \frac{1}{2}} = \boldsymbol{6x^3 - 4x^2 + 2x - 10 + \frac{4}{x - \frac{1}{2}}}$$

But we **need the factor of 2** that is missing**.**

$$\frac{6x^4 - 7x^3 + 4x^2 - 11x + 9}{\boldsymbol{2}\left(x - \frac{1}{2}\right)} = \frac{6x^3 - 4x^2 + 2x - 10}{\boldsymbol{2}} + \frac{4}{\boldsymbol{2}\left(x - \frac{1}{2}\right)}$$

$$\frac{6x^4 - 7x^3 + 4x^2 - 11x + 9}{2x - 1} = \boldsymbol{3x^3 - 2x^2 + x - 5 + \frac{4}{2x - 1}}$$

Do not forget the 'missing factor', it is an integral part to having the correct solution!

- Next, we will see an example where we have **factors in a quadratic form**
- if the **factor can be factored**, we run the **Synthetic Division Multiple times, like going down steps**.
- See the example on the following page

Example 7: Divide $x^4 + 9x^3 - 5x^2 - 36x + 4 \ by \ x^2 - 4$

Solution 7: Synthetic Division does not work with a factor that is a quadratic, but $x^2 - 4$ is a Difference of Squares and can be factored to $(x + 2)(x - 2)$. So, we have two roots, $x = 2, -2$. Start with either one then use your solution from the first one and run Synthetic Division again with the remaining factor.

Start with $x = 2$

2	1	9	-5	-36	4	$Row\ 1$
		2	22	34	-4	$Row\ 2$
	1	11	17	-2	0	$Row\ 3$

We would expect this because there is another factor

Run it again using the Solution above and the other factor

-2	1	11	17	-2
		-2	-18	2
	1	9	-1	0

So, the result is:

$$\frac{x^4 + 9x^3 - 5x^2 - 36x + 4}{x^2 - 4} = \frac{\boldsymbol{(x - 2)(x + 2)(x^2 + 9x - 1)}}{\boldsymbol{(x - 2)(x + 2)}} = \boldsymbol{x^2 + 9x - 1}$$

Section 3.3 – Practice Problems

1. Compute the quotient using long division. Write answer in two ways:

   i) $Dividend\ =\ (quotient)(Divisor) + Remainder$

   ii) $\dfrac{dividend}{divisor} = (quotient)\dfrac{remainder}{divisor}$

a) $x - 3\ \overline{)\ x^3 - 8x^2 - 3x + 2}$

b) $\dfrac{8x^3 - 1}{2x - 1}$

c) $x^2 + 1\ \overline{)\ x^5 + 2x^4 - x^3 + x^2 - 3x + 4}$

d) $\dfrac{x^4 - 3x^2 + 8}{x^2 - 1}$

e) $x^2 - 4x - 12 \overline{)x^3 + 2x^2 - 13x + 10}$

f) $\dfrac{x^3 - 5x + 1}{x^2 - 2x}$

g) $x^3 + 3x + 2 \overline{)x^4 + 6x^3 + 11x^2 + 6x}$

h) $\dfrac{x^4 + 9x^3 - 5x^2 - 32x + 3}{x^2 - 1}$

2. Use synthetic division to find the quotient $Q(x)$ and the remainder R when the polynomial $P(x)$ is divided by the given binomial.

a) $P(x) = x^3 + 2x^2 - 3x + 1; x - 2$

b) $P(x) = x^3 - a^3; x - a$

c) $P(x) = 4x^3 + 5x - 3; x + 2$

d) $P(x) = x^5 - 5x^3 + 10; x - 1$

e) $P(x) = 0.1x^2 + 0.2; x - 2.1$

f) $P(x) = x^5 + 1; x + 1$

g) $P(x) = 3x^4 + x^3 - 3x + 1; 3x + 1$

h) $P(x) = 2x^4 - x^3 + 2x - 1; 2x - 1$

i) $P(x) = 3x^5 + 2x^4 + 5x^3 - 7x + 3;$
$x + 0.8$

j) $P(x) = 3x^4 - 3x^3 + 2x^2 - 3x + 1;$
$x - 0.4$

k) $P(x) = x^4 - 5x^3 - 4x^2 + 5x + 3;$
$x^2 - 1$

l) $P(x) = x^5 - x^4 - 8x^3 + 7x^2 + 7x - 30;$
$x^2 - x - 6$

3. Divide by synthetic division. Write answers in the form $f(x) = c(x)g(x) + r$ where $f(x)$ is the given polynomial and $c(x)$ is the given factor.

a) $4x^3 - 7x^2 - 11x + 5; x + 2$

b) $6x^3 - 16x^2 + 17x - 6; 3x - 2$

c) $x^3 - 64;\ x - 2$

d) $4x^3 + 16x^2 - 23x + 15; 2x - 1$

e) $x^3 - 4x; x - 1 + \sqrt{3}$

f) $-3x^3 + 8x^2 + 10x - 8; x - 2 - \sqrt{2}$

g) $x^4 - 4x^3 - 15x^2 + 58x - 40; x - 5$

h) $x^5 - 2x^4 + x^3 - 5; x - 2$

i) $x^4 + 6x^3 + 11x^2 + 6x; x^2 + 3x + 2$

j) $x^4 + 9x^3 - 5x^2 - 36x + 4; \quad x^2 - 4$

k) $x^{3n} + 16x^{2n} + 64x^n + 64; \quad x^n + 4$
$(n\ is\ a\ positive\ integer)$

l) $x^{3n} - 9x^{2n} + 27x^n - 27; \quad x^n - 3$

4. Use synthetic division to solve for $k\ and\ m$

a) When $x^3 + kx + 1$ is divided by $x - 2$ the remainder is -3

b) When $x^3 - x^2 + kx - 8$ is divided by $x - 4$ the remainder is 0

c) When $2x^4 + kx^2 - 3x + 5$ is divided by $x - 2$ the remainder is 3

d) When $x^3 + kx + 6$ is divided by $x + 2$ the remainder is 4

e) When $x^3 + kx^2 - 2x - 7$ is divided by $x + 1$ the remainder is 5. What is the remainder when it is divided by $x - 1$?

f) When $kx^3 + mx^2 + x - 2$ is divided by $x - 1$, the remainder is 6. When the polynomial is divided by $x + 2$, the remainder is 12.

g) $x^4 + kx^3 - mx + 15$ has no remainder when divided by $x - 1$ and $x + 3$

h) When $P(x) = 3x^4 + kx^2 + 7$ is divided by $x - 1$, the remainder is the same as when $f(x) = x^4 + kx - 4$ is divided by $x - 2$.

See Website for Detailed Answer Key

Extra Work Space