Section 3.1 – Polynomials

- We have seen Polynomials many times up to this point, but now we finally put it all together

Definition of a Polynomial

Let $a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1, a_0$ be real numbers, and n a whole number.

A Polynomial is an expression of the form:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad with\ a_n \neq 0.$$

The Polynomial is of $\boldsymbol{degree\ n}$, with $\boldsymbol{a_n\ the\ leading\ coefficient}$

- We have mentioned Quadratics repeatedly, it is a Polynomial with a special name, there are more:

Polynomials in Standard Form	Degree	Leading Coefficient	Special Name
$f(x) = 7x^4 + 5x^3 - 2x + 6$	4	7	$Quartic$
$h(x) = -2x^3 + 5x^2 - 8x - 7$	3	-2	$Cubic$
$g(x) = \sqrt{3}x^2 - x + 1$	2	$\sqrt{3}$	$Quadratic$
$k(x) = 5x + 6$	1	5	$Linear$
$r(x) = 6$	0	6	$Constant$

- Remember that to be classified as a Polynomial you must have:

A **WHOLE NUMBER EXPONENT** on the **VARIABLES**

The **CONSTANTS** are **REAL NUMBERS**

Examples:

$f(x) = 3x^{-1} + 2x - 5,$ is NOT a POLYNOMIAL, the exponent on the variables is not a whole number

$g(x) = \sqrt{2}x^3 + \sqrt{-3}x,$ is NOT a POLYNOMIAL, the coefficient $\sqrt{-3}$ is not real.

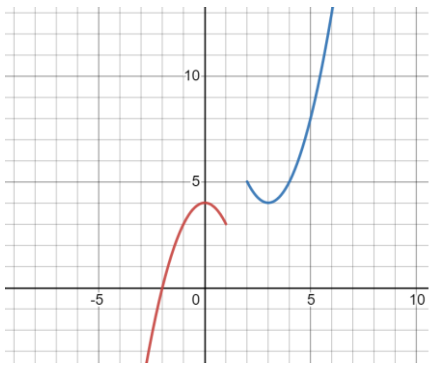
$h(x) = \frac{2x-3}{x^2},$ is NOT a POLYNOMIAL, the x^2 in the denominator is an x^{-2} in the numerator

$m(x) = 3x - 7x^{\frac{1}{2}},$ is NOT a POLYNOMIAL, the exponent of the variable is not a whole number

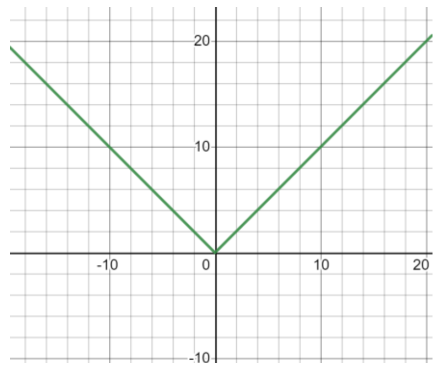
Shape of Polynomial Graphs

- Polynomials are what we call CONTINOUS GRAPHS, they do not have breaks, corners, or sharp edges
- You can graph a Polynomial without lifting your pencil off the paper.

Example of NON-POLYNOMIAL graphs

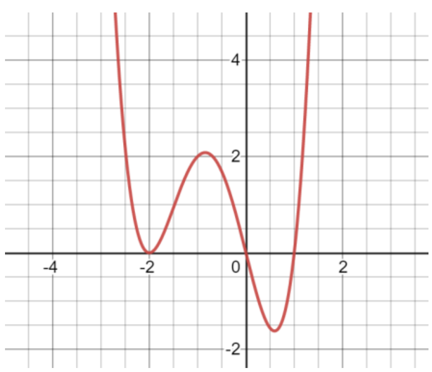


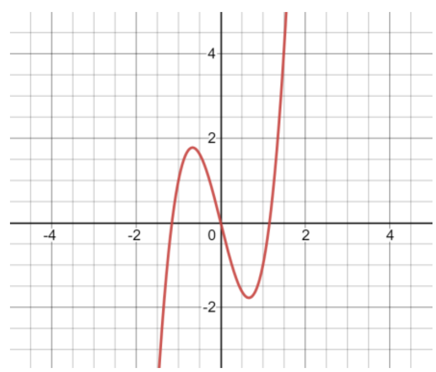
This graph has a break in it.



This graph has a sharp corner.

Example of POLYNOMIAL graphs





Polynomials of the form $f(x) = x^n$ $\quad$ *and* their reflections in the $x - axis$ $f(x) = -x^n$

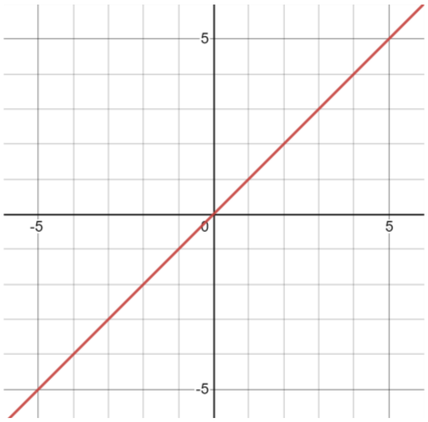
This is the start of examining the end-behaviour of polynomials.

- What we are examining is a reflection in the $x - axis$
- A sign change of the $y - values$ of the original graph
- You will start to see a pattern to this and we can identify a relationship between graph with a negative first term or a positive first term

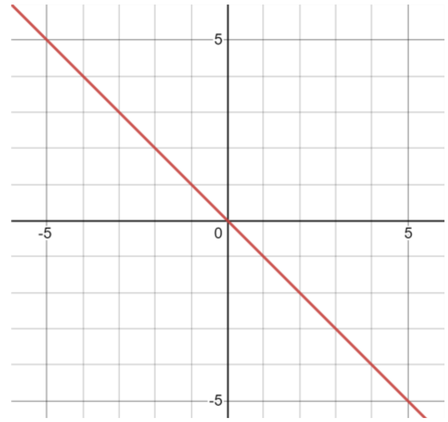
See the examples on the next page

Examples

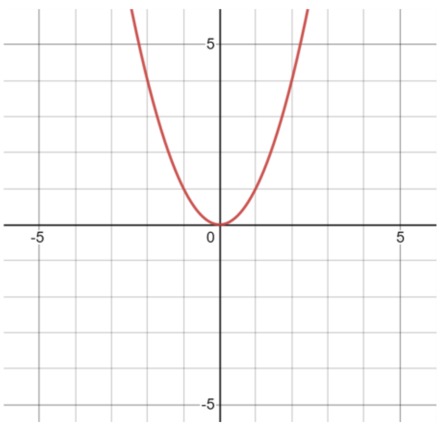
$f(x) = x$



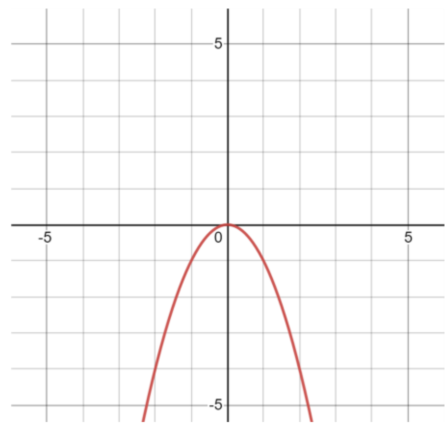
$f(x) = -x$



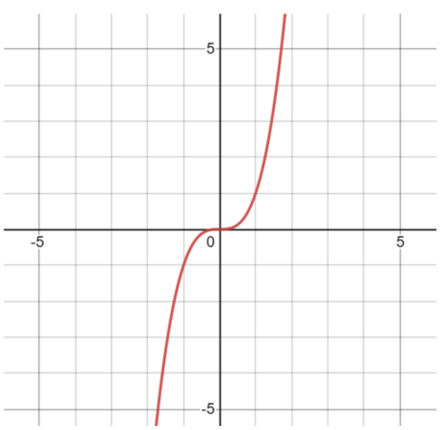
$f(x) = x^2$



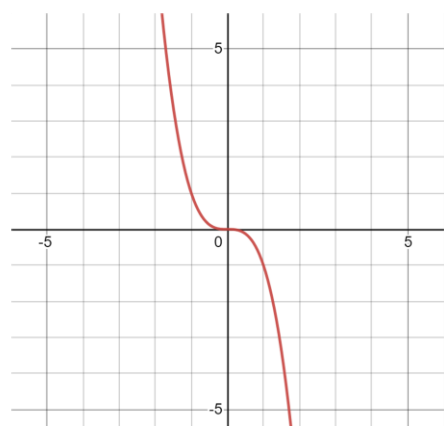
$f(x) = -x^2$



$f(x) = x^3$



$f(x) = -x^3$



Comparing End-Behaviour of Functions

- When we say 'end-behaviour', what we mean is:
  "What happens to the graphs as x gets infinitely large or small?"
- This is expressed as:

$$x \to \pm\infty, \quad f(x) \to ?$$

As x gets infinitely large or small, y values become what?

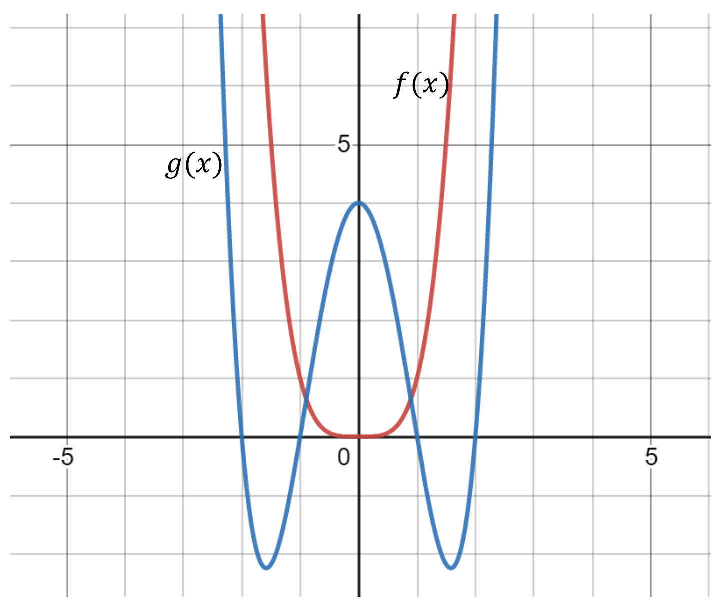
Consider this…

$f(x) =$	x^n n *is odd*	$-x^n$ n *is odd*	x^n n *is even*	$-x^n$ n *is even*
Domain ***Range***	***All Real #'s*** ***All Real #'s***	***All Real #'s*** ***All Real #'s***	***All Real #'s*** $y \geq 0$	***All Real #'s*** $y \leq 0$
$x \to \infty$	$f(x)$ ***increases*** $f(x) \to \infty$	$f(x)$ ***decreases*** $f(x) \to -\infty$	$f(x)$ ***increases*** $f(x) \to \infty$	$f(x)$ ***decreases*** $f(x) \to -\infty$
$x \to -\infty$	$f(x)$ ***decreases*** $f(x) \to -\infty$	$f(x)$ ***increases*** $f(x) \to \infty$	$f(x)$ ***increases*** $f(x) \to \infty$	$f(x)$ ***decreases*** $f(x) \to -\infty$

One-term function and **multi-term function**, **<u>do not change</u>** in their **end-behaviour**, all that matters is the leading term! Let's see that as an example.

If $f(x) = x^4$ ***and*** $g(x) = x^4 - 5x^2 + 4$

x	$f(x)$	$g(x)$
0	0	4
2	16	0
-2	16	0
5	625	504
-5	625	504
10	10 000	9504
-10	10 000	9504
100	10^8	$9.99 \cdot 10^7$
-100	10^8	$9.99 \cdot 10^7$

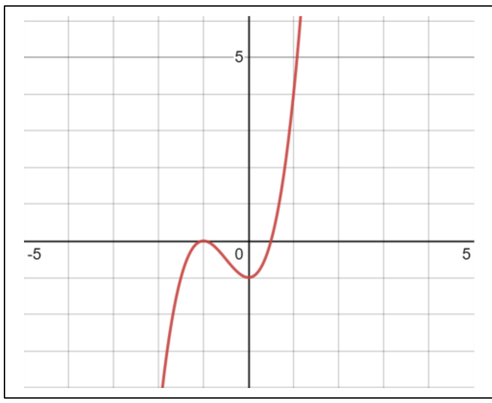


Degree and Leading Coefficient of Polynomial Functions

- The **main contributors** to the change in **end-behaviour** is the **leading term:** $a_n x^n$
- **The leading coefficient a_n** will give you the **end-behaviour direction**
- **The exponent n** will tell you **the shape**

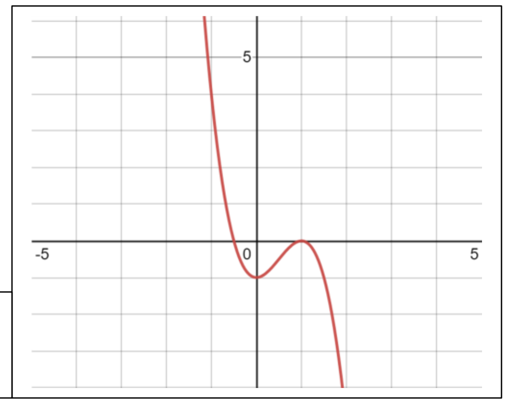
If the **degree** of the Polynomial **is odd:** The **ends will be in opposite directions**

- **A Positive Leading Coefficient (a): Down on the Left, Up on the Right**
- **A Negative Leading Coefficient $(-a)$: Down on the Right, Up on the Left**



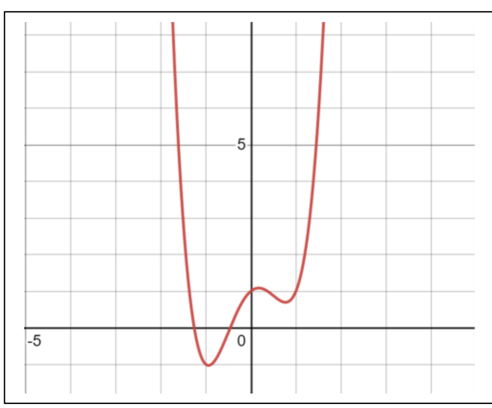
$2x^3 + \cdots$

$-2x^3 + \cdots$



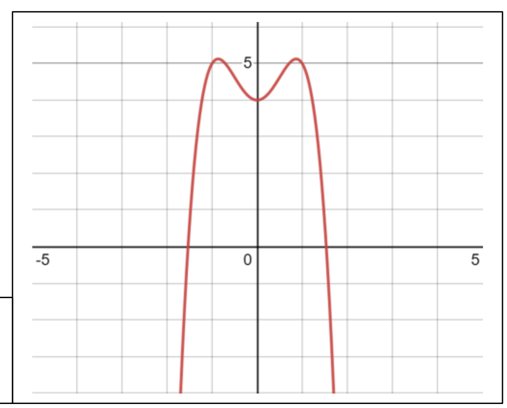
If the **degree** of the Polynomial **is even:** The **ends will be in same direction**

- **A Positive Leading Coefficient (a): Up on the Left, Up on the Right**
- **A Negative Leading Coefficient $(-a)$: Down on the Right, Down on the Left**



$2x^4 + \cdots$

$-2x^4 + \cdots$



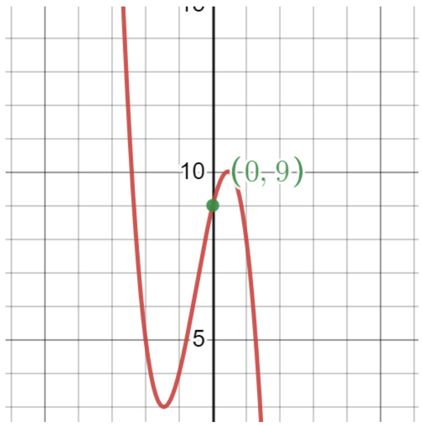
Constant Value of a Polynomial

- **The constant is an important term, it is the one without a variable.**
- **Why so important?**
- **Because when $x = 0$, we have the $y - intercept$**

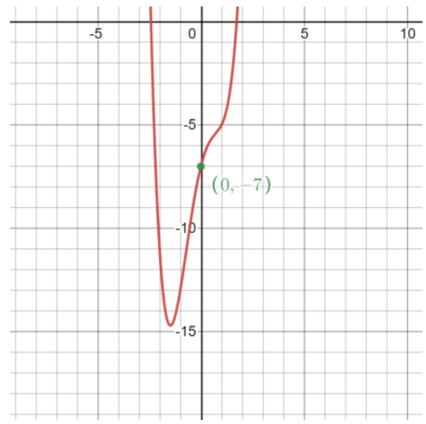
$$\boldsymbol{The\ Constant\ is\ the\ y - intercept}$$

Consider…

$\boldsymbol{y = -2x^3 + \cdots + 9}$



$\boldsymbol{y = x^4 + \cdots - 7}$



Examples: Find the $y - intercept$ of the following Polynomial Functions

a) $f(x) = -2(x+1)^3$ b) $g(x) = (2x-1)(x+4)(x-3)$ c) $h(x) = x^4 - 5$

Solutions: Remember, we get the $y - intercvept$, when $x = 0$

a) $f(x) = -2(x+1)^3 \quad \rightarrow \quad f(0) = -2(0+1)^3 \quad \rightarrow \quad f(0) = -2$

$$y - int\ is\ (0, -2)$$

b) $g(x) = (2x-1)(x+4)(x-3) \quad \rightarrow \quad g(x) = (2(0)-1)(0+4)(0-3) \quad \rightarrow \quad g(0) = 12$

$$y - int\ is\ (0, 12)$$

c) $h(x) = x^4 - 5 \quad \rightarrow \quad h(0) = 0^4 - 5 \quad \rightarrow \quad h(0) = -5$

$$y - int\ is\ (0, -5)$$

Zeros of a Polynomial Function

Recall from Pre-Calculus 11 that the 'Zeros of a Polynomial' are the:

- $x - intercepts$
- Solutions
- Roots

All the Same Thing

Example: Find the Real Zeros of: a) $f(x) = -3x^4 + 3x^2$ b) $g(x) = x^3 - 2x^2 + x - 2$

Solution:

a) $f(x) = -3x^4 + 3x^2 \quad \rightarrow \quad f(x) = -3x^2(x^2 - 1)$ Factor out GCF

$\rightarrow \quad f(x) = -3x^2(x+1)(x-1)$ Difference of Squares

Solve for the factor when $f(x) = 0$, $\boldsymbol{so}$: $\boldsymbol{-3x^2 = 0}$, $\boldsymbol{(x+1) = 0}$, $\boldsymbol{(x-1) = 0}$

$$\boldsymbol{x = 0, 1, -1}$$

b) $g(x) = x^3 - 2x^2 + x - 2 \quad \rightarrow \quad g(x) = (x^3 - 2x^2)(+x - 2)$

$\rightarrow \quad g(x) = x^2(x-2) + 1(x-2)$

$\rightarrow \quad \boldsymbol{g(x)} = (x^2 + 1)(x - 2)$

Solve for the factor when $g(x) = 0$, $\boldsymbol{so}$: $\boldsymbol{x^2 + 1 = 0}$ $\boldsymbol{(Has\ No\ Solution)}$, $\boldsymbol{(x-2) = 0}$

$$\boldsymbol{x = 2}$$

Zeros and Turning Points of a Degree n Polynomial

- A Polynomial of $\boldsymbol{degree\ n}$, has **at most**, $\boldsymbol{n\ real\ zeros}$
- An n^{th} degree polynomial, with $\boldsymbol{n\ an\ even\ number}$, can **intersect the $\boldsymbol{x - axis}$** anywhere from $\boldsymbol{0\ to\ n\ times}$

  Remember, Even Degree Polynomials start and end both up or down, so do not have to cross the $x - axis$

- An n^{th} degree polynomial, with $\boldsymbol{n\ an\ odd\ number}$, can **intersect the $\boldsymbol{x - axis}$** anywhere from $\boldsymbol{1\ to\ n\ times}$

  Remember, Odd Degree Polynomials start and end in opposite directions, so do have to cross the $x - axis$

Example: What is the minimum and maximum number of intersections of the $x - axis$ for the following polynomial functions?

a) $y = -9x^5 + \cdots$

b) $y = 5x^4 + \cdots$

Solution:

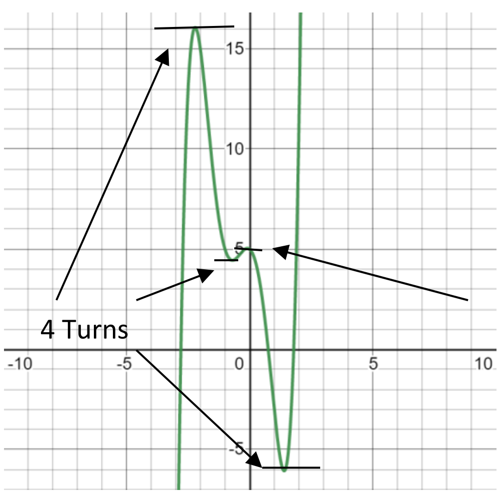
a) Since it is an $\boldsymbol{odd\ n\ value, minimum\ is\ 1}$ **and** $\boldsymbol{maximum\ is\ n}$ **(so 5)**
b) Since it is an $\boldsymbol{even\ n\ value}$, $\boldsymbol{minimum\ is\ 0}$ **and** $\boldsymbol{maximum\ is\ n}$ **(so 4)**

Turning Points

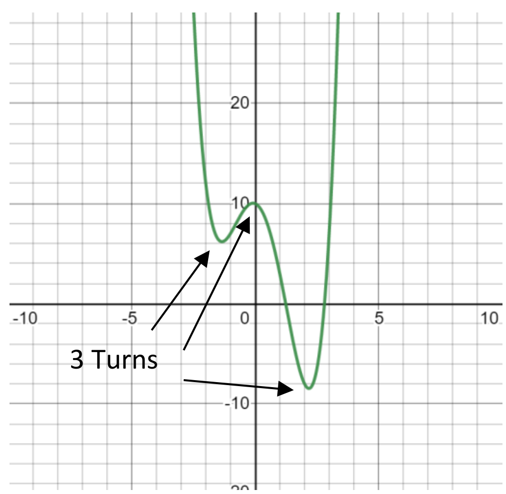
- If a polynomial has $\boldsymbol{n\ turning\ points}$, it is of **minimum degree:** $(\boldsymbol{n+1})$
- In other words**:**

$\boldsymbol{The\ number\ of\ turning\ points}$ of a graph **is** $\boldsymbol{one\ less\ than\ its\ degree}$

Minimum Degree of 5



Minimum Degree of 4



Multiplicity of a Polynomial

- A polynomial of degree n, had at most, n distinct solutions.
- But if a solution repeats itself, we say that the solution has multiplicity of the number of times it is repeated.
- If you end up with the same root, or a factor to a power, we have that multiplicity of the solution

$f(x) = x^4 + x^3 - 6x^2$ Degree 4, 0 to 4 roots

$0 = x^2(x^2 + x - 6)$

$0 = x^2(x+3)(x-2)$ Factor Completely

$x^2\ has\ zero\ at\ 0, mulitiplicity\ 2$

$(x+3)\ has\ zero\ at\ -3, multiplicity\ 1$

$(x-2)\ has\ zero\ at\ 2, multiplicity\ 1$

Section 3.1 – Practice Problems

1. Suppose $y = f(x)$ has the point (a, b). Write (a, b) with the transformations described.

a) A polynomial is _______________, because it has no gaps, breaks, of holes.

b) A polynomial of **even degree** has at most how many zeros? At least how many zeros? How many turning points?

c) A polynomial of **odd degree** has at most how many zeros? At least how many zeros? How many turning points?

d) If $x = a$ is a zero of a polynomial then:

$x = a$ is also called a _________

A factor of the polynomial is: _________

$(a, 0)$ is what kind of intercept?

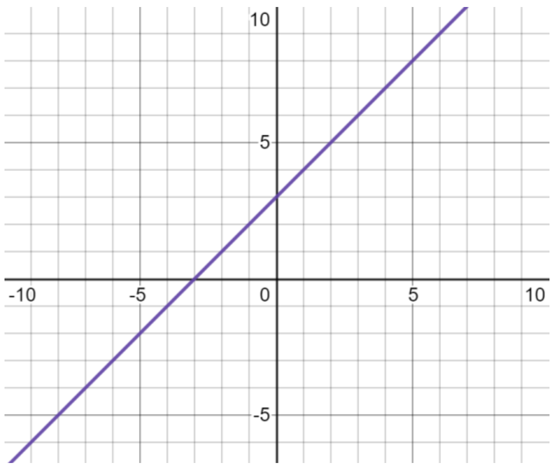
e) A polynomial written in Standard Form is written with powers on the variables in what kind of order?

2. Are the following equations polynomials? If so, state the degree, leading coefficient, and special name (if available), if it is not a polynomial, state the reason.

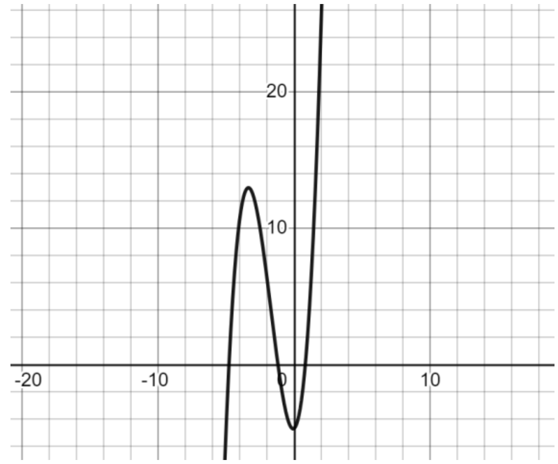
Equation	Polynomial Y/N	Degree	Leading Coefficient	Special Name
a) $-2x^3 + x^2 - 5$				
b) $\sqrt{2}x^4 - \sqrt{3x} + 2$				
c) $-\frac{1}{3}x^2 + \sqrt{-2}x + 1$				
d) $3x + 2$				
e) 5				

3. Which of the following graphs are Polynomial? If they are not, explain why.

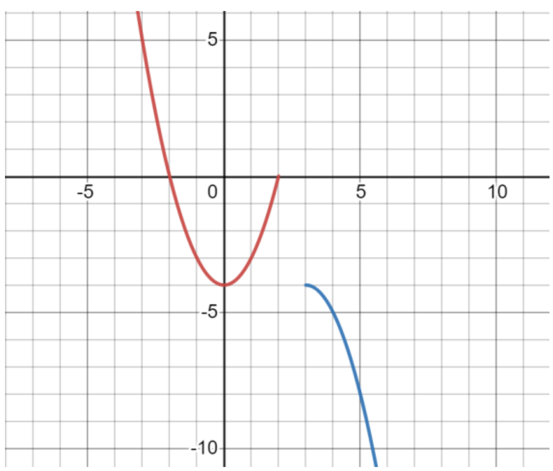
a)



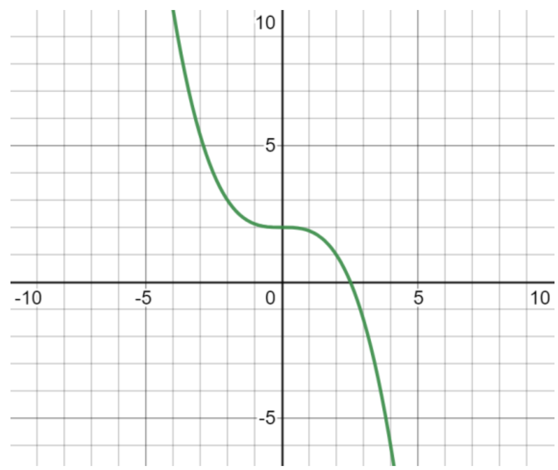
b)



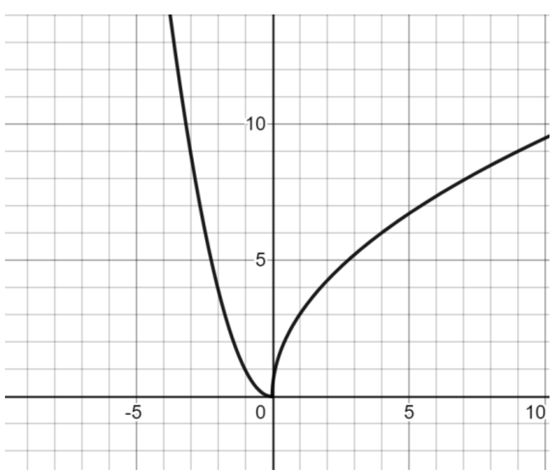
c)



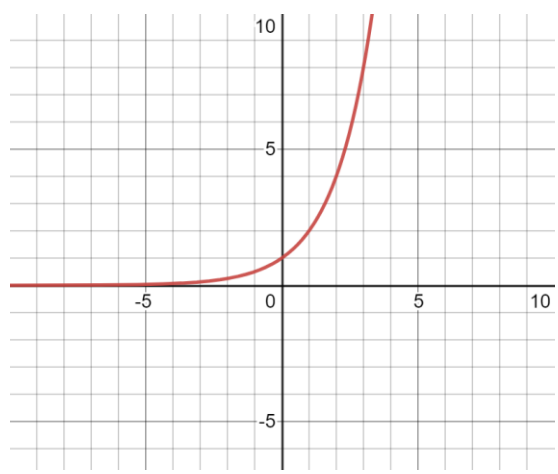
d)



e)

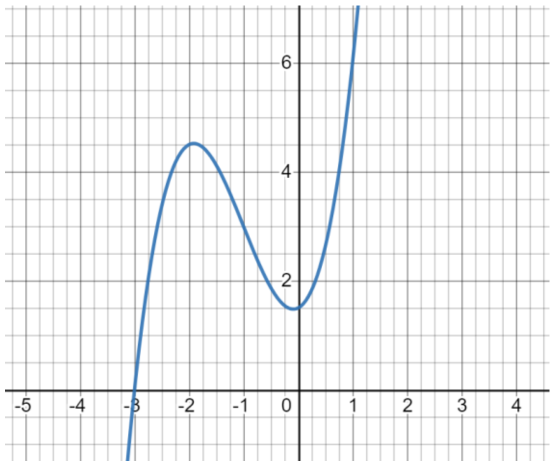


f)

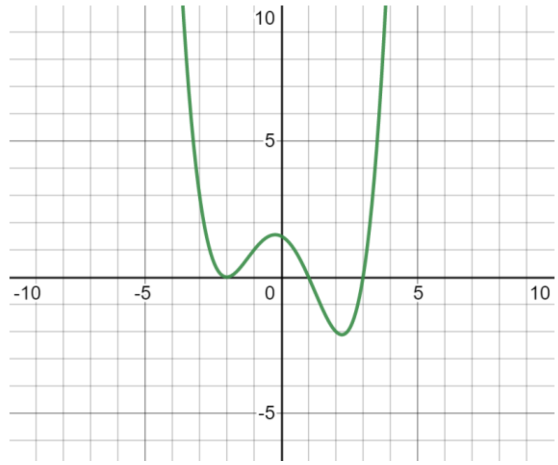


4. What are the real zeros of the following polynomials?

a)



b)



5. What is the max/min number of zeros of the following Polynomials.

a) $3x^5 + 3x^4 - 2x^2 + 1 = 0$

$8x^6 + 3x^4 - 2x^2 + 1 = 0$

b) Given a general polynomial:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

What is the max/min number of roots when n is odd? ________

What is the max/min number of roots when n is even? ________

6. State whether the following are polynomial functions. If so, what is the degree, if not why?

a) $f(x) = -x^4 + 4x^4 + 2$	b) $f(x) = \sqrt{x}$
c) $f(x) = \frac{1}{x}$	d) $f(x) = 0$
e) $f(x) = (x - 2)^3$	f) $f(x) = (x + 1)^{-2}$
g) $f(x) = x^3 - \sqrt{2}x + \frac{1}{3}$	h) $f(x) = 2^{-3}x^2$
i) $f(x) = \sqrt{2}x^2$	j) $f(x) = \frac{1}{x + 1}$

7. Determine the end behaviour of the polynomials below.

a) $f(x) = 3x$	b) $f(x) = -3x$
c) $f(x) = 2x + 3x^2$	d) $f(x) = 2x - 3x^2$
e) $f(x) = -2x + x^2 + 3x^3$	f) $f(x) = 2x - x^2 - 3x^3$

g) $f(x) = 3x^4 - x^2 + 1$

h) $f(x) = -3x^4 + x^2 - 1$

i) $f(x) = x^4 + 2x^3 + x^5 - 2$

j) $f(x) = x^4 - 2x^3 - x^5 + 2$

8. Find a function in the form $y = cx^n$ that has the same end behaviour as the given function.

a) $f(x) = -3x^3 - 2x^2 + 1$

b) $g(x) = 2x^3 + x^2 - 1$

c) $h(x) = 2.3x^4 - 4x^2 + 6x$

d) $k(x) = -2.4x^5 + 3x^4 - 2x - 1$

9. Find all the real zeros, and the multiplicity of each zero.

a) $f(x) = x^2 - 4$

b) $f(x) = (x - 4)^2$

c) $g(x) = x^3 - 4x^2 + 4x$

d) $g(x) = 2x(x^2 - 2x - 1)$

e) $h(x) = x^4 - x^3 - 20x^2$

f) $h(x) = \frac{1}{3}x^4 - \frac{1}{3}$

g) $k(x) = x^4 + 3x^2 + 2$

h) $k(x) = x^3 - 4x^2 - 25x + 100$

i) $l(x) = -x^3 - 3x^2 + 4x + 12$

j) $l(x) = x^3 - 5x^2 - x + 5$

k) $m(x) = x^4 - 2x^2 + 1$

l) $m(x) = -x^4 + 3x^2 - 2$

m) $n(x) = -x^4 + 4x^3 - 4x^2$

n) $n(x) = -x^2(x^2 - 1) + 4(x^2 - 1)$

See Website for Detailed Answer Key of the Remainder of the Questions

Extra Work Space