Section 2.6 – Combined Transformations

- We can summarize the transformation steps as follows:

$$y = f(x) \quad versus \quad y = af[b(x \pm c)] + d$$

Recall:

Transforming Vertically

- ✓ $\boldsymbol{a > 1}$ is a **vertical expansion** by a $\boldsymbol{factor\ of\ a}$
- ✓ $\boldsymbol{0 < a < 1}$ is a **vertical compression** by a $\boldsymbol{factor\ of\ a}$
- ✓ $\boldsymbol{a < 0}$ **(negative),** is a **reflection in the** $\boldsymbol{x - axis}$ $(the\ y - values\ change\ sign)$
- ✓ $\boldsymbol{+d}$ **shifts up** $\boldsymbol{d\ units}$
- ✓ $\boldsymbol{-d}$ **shifts down** $\boldsymbol{d\ units}$

Transforming Horizontally

- ✓ $\boldsymbol{b > 1}$ is a **horizontal compression by** $\boldsymbol{a\ factor\ of\ \frac{1}{b}}$
- ✓ $\boldsymbol{0 < b < 1}$ is a **horizontal expansion** $\boldsymbol{by\ a\ factor\ of\ \frac{1}{b}}$**, and since** $\boldsymbol{b}$ **is a fraction,** $\boldsymbol{\frac{1}{b} = b}$
- ✓ $\boldsymbol{b < 0}$ **(negative),** is a **reflection in the** $\boldsymbol{y - axis}$ $(the\ x - values\ change\ sign)$
- ✓ $\boldsymbol{+c}$ **shifts left** $\boldsymbol{c\ units}$ $(\boldsymbol{c > 0})$
- ✓ $\boldsymbol{-c}$ **shifts right** $\boldsymbol{c\ units}$ $(\boldsymbol{c > 0})$

REFLECTIONS/COMPRESIONS/EXPANSIONS ALWAYS COME FIRST – THEN ANY TRANSLATIONS

Example 1: $y = f(x)$ transformed to $y = -2f[3(x + 5)] - 7$

If $(6, -3)$ is on $f(x)$ how does the point transform?

Solution 1:

Vertically

- Refection in the $x - axis$ (multiply $y - values\ by - 1$)
- Expansion by $a\ factor\ of\ 2\ (multiply\ y - value\ by\ 2)$
- Translation $down\ 7\ (subtract\ 7\ from\ y - value)$

Horizontally

- Compression by a factor of $\frac{1}{3}$ (multiply $x - value$ by $\frac{1}{3}$)
- Translation 5 $left\ (subtract\ 5\ from\ the\ x - value)$

So

$$(6, -3)$$

Transforms to:

$$\left(6\left(\frac{1}{3}\right) - 5, -3(-1)(2) - 7\right)$$

$$(2 - 5, 6 - 7)$$

$$(-3, -1)$$

- It is necessary to factor out the $b\ term$ from any included horizontal translation

Example 2: $\boldsymbol{y = f(2x - 6)}$

Not allowed – factor out the 2

Solution 2:

$\boldsymbol{y = f(2x - 6)} \quad \rightarrow \quad \boldsymbol{y = f[2(x - 3)]}$

- Horizontal Compression by a factor of $\frac{1}{2}$
- Then 3 units to the right

Solving Combined Operations

There are two methods of solving combined equations:

1. A step-by-step approach
2. A one-shot calculation with the corresponding coordinates

Example 3: If the point $(3, 2)$ is on the graph $y = f(x)$, what point is on $y = -4f(6 - 3x) + 1$?

Solution 3:

First re-write the new function in the usual form with the $b\ term$ factored out

$$y = -4f(6 - 3x) + 1 \quad \rightarrow \quad y = -4f[-3(x - 2)] + 1$$

Method 1:

- -4 reflects points about the $x - axis$ with a vertical expansion by a $factor\ of$ 4 so, $(3, 2) \rightarrow (3, -8)$
- -3 reflects points about the $y - axis$ with a horizontal compression by a $factor\ of$ $\frac{1}{3}$ so, $(3, -8) \rightarrow (-1, -8)$
- $x - 2$ shifts the point two units horizontally to the right, so $(-1, -8) \rightarrow (1, -8)$
- $+1$ shifts the point one unit vertically up, so $(1, -8) \rightarrow (1, -7)$
- So, the transformation is:

$$(3, 2) \rightarrow (1, -7)$$

Method 2

If $y = f(x)$ has a point (m, n), then

$y = af[b(x - c)] + d$ has a point:

$$\left(\frac{m}{b} + c, an + d\right)$$

$$a = -4, b = -3, c = 2, d = 1$$

So,

$$\left(\frac{3}{-3} + 2, (-4)(2) + 1\right)$$

$$(1, -7)$$

Example 4: If the point $(-1, 2)$ is on the graph $y = f^{-1}(x)$, what point is on $y = -3f(8 + 2x) - 1$?

Solution 4:

First re-write the new function in the usual form with the $b\ term$ factored out:

$$y = -3f(8 + 2x) - 1 \quad \rightarrow \quad y = -3f[2(x + 4)] - 1$$

Method 1:

- If $(-1, 2)$ is on $y = f^{-1}(x)$, then $(2, -1)$ is on $f(x)$
- -3 reflects points about the $x - axis$ with a vertical expansion by a $factor\ of$ 3 so, $(2, -1) \rightarrow (2, 3)$
- 2 is a horizontal compression by a $factor\ of\ \frac{1}{2}$ so, $(2, 3) \rightarrow (1, 3)$
- $x + 4$ shifts the point four units horizontally to the left, so $(1, 3) \rightarrow (-3, 3)$
- -1 shifts the point one unit vertically up, so $(-3, 3) \rightarrow (-3, 2)$
- So, the transformation is:
  $$(-1, 2) \rightarrow (-3, 2)$$

Method 2

$y = f^{-1}(x)$ has a point (n, m), then

$y = af[b(x - c)] + d$ has a point:

$$\left(\frac{m}{b} + c, an + d\right)$$

$$a = -3, b = 2, c = -4, d = -1$$

So,

$$\left(\frac{2}{2} + (-4), (-3)(-1) - 1\right)$$

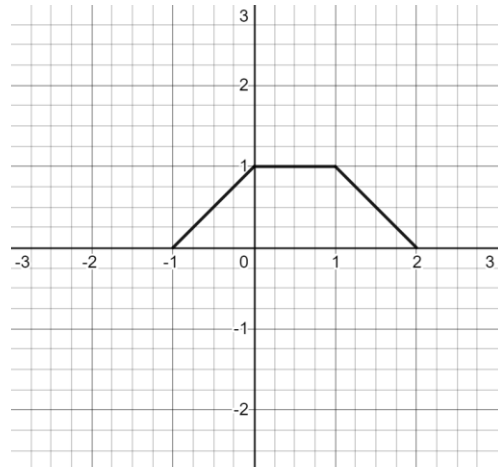
$$(-3, 2)$$

Transforming Graphs

- Using a step-by-step approach for graph transformations can be tedious
- It is helpful to pick a number of reference points, transforming them, and re-drawing the graph in its entirety.
- See the example below

Example 4: Given the graph $y = f(x)$ below, graph $y = -2f[-2(x+1)] + 1$

Solution 4:



Remember:

If $y = f(x)$ has point (m, n)

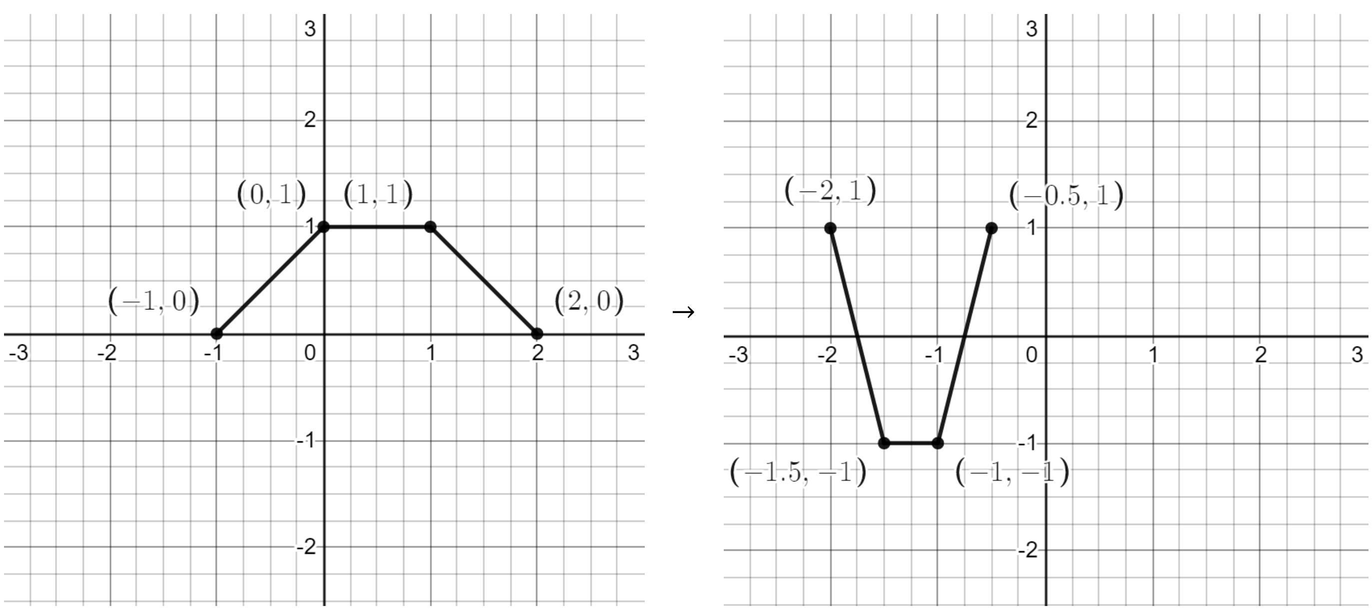
$y = af[b(x-c)] + d$ has the point:

$$\left(\frac{m}{b} + c, an + d\right)$$

$$a = -2, b = -2, c = -1, d = 1$$

Use the reference points to calculate the transformations: $(-1,0), (0,1), (1,1), (2,0)$

$(-1,0) \rightarrow \left(\frac{-1}{-2} - 1, (-2)(0) + 1\right) \rightarrow \left(-\frac{1}{2}, 1\right)$	$(0,1) \rightarrow \left(\frac{0}{-2} - 1, (-2)(1) + 1\right) \rightarrow (-1,-1)$
$(1,1) \rightarrow \left(\frac{1}{-2} - 1, (-2)(1) + 1\right) \rightarrow \left(-\frac{3}{2}, -1\right)$	$(2,0) \rightarrow \left(\frac{2}{-2} - 1, (-2)(0) + 1\right) \rightarrow (-2,1)$



Section 2.6 – Practice Problems

1. Suppose $y = f(x)$ has the point (a, b). Write (a, b) with the transformations described.

a) $y = f(x-1)+1$	b) $y = f(1-x)$
c) $y = -f(-x)$	d) $y = f(x)+1$
e) $y = f(-x)$	f) $y = -f(x)$
g) $y = f(x+1)$	h) $y = f^{-1}(x)$
i) $y = -f^{-1}(x)$	j) $y = f^{-1}(x)+1$
k) $y = f^{-1}(x-1)$	l) $y = f^{-1}(-x)+1$
m) $y = f^{-1}(x)+1$	n) $y = -f^{-1}(-x)+1$

2. If points $(4, -2)$ and (a, b) are on the graph of $y = f(x)$, what points must be on the following graphs?

a) $y = f(x-1) - 3$

b) $y = -f(-x) + 1$

c) $y = -f(x+2) - 1$

d) $y = |f(2x)|$

e) $y = \frac{1}{2}f(x-1) + 4$

f) $y = -|f(x-2)|$

g) $y = f\left(-\frac{1}{2}x\right) + 1$

h) $y = -f(1-x)$

i) $y = f^{-1}(x) + 2$

j) $y = f^{-1}(x + 1)$

3. If $f(x) = x^2 - 1$, determine the equation after each of the following transformations.

a) $y = f(x + 2)$

b) $y = f\left(\frac{1}{2}x\right) + 1$

c) $y = -f(x - 1) + 2$

d) $y = 2f(1 - x) + 3$

e) Expand vertically by a factor of 3

f) Expand horizontally by a factor of 3

4. If $4x^2 + y^2 = 36$, determine the equation after each of the following transformations (these are not intuitive, is it in the form $y = f(x)$?

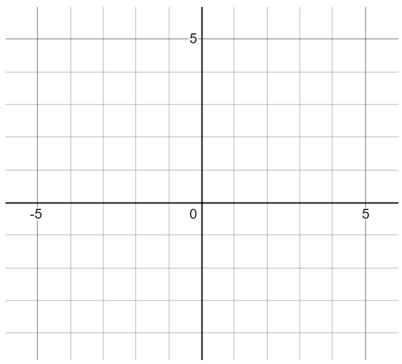
a) Expand horizontally by a factor of 2

b) Compress vertically by a factor of $\frac{1}{3}$

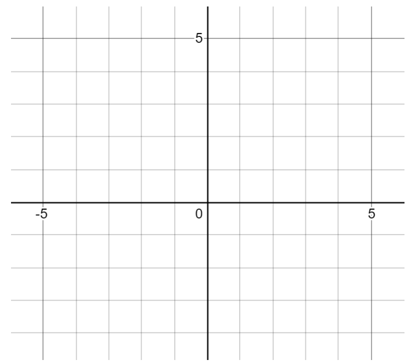
c) Compress horizontally by a factor of $\frac{1}{2}$ and expand vertically by a factor of $\frac{4}{3}$

5. Write an expression for $f(x)$ obtained by reflecting the graph $g(x) = \frac{1}{2}x - 2$, about the: Drawings may help.

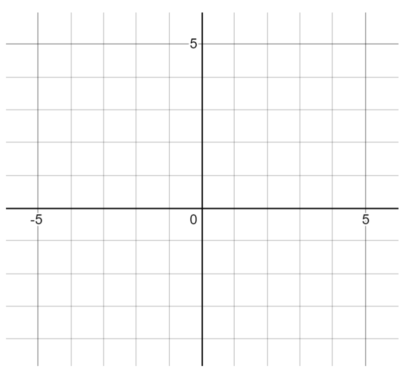
a) $x - axis$



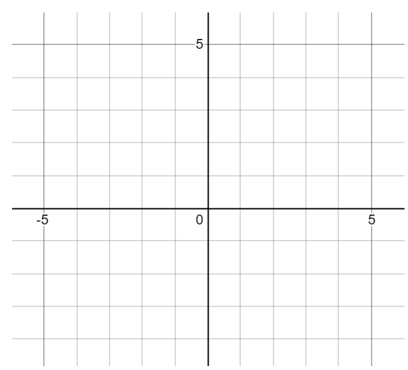
b) $y - axis$



c) $line\ x = 2$

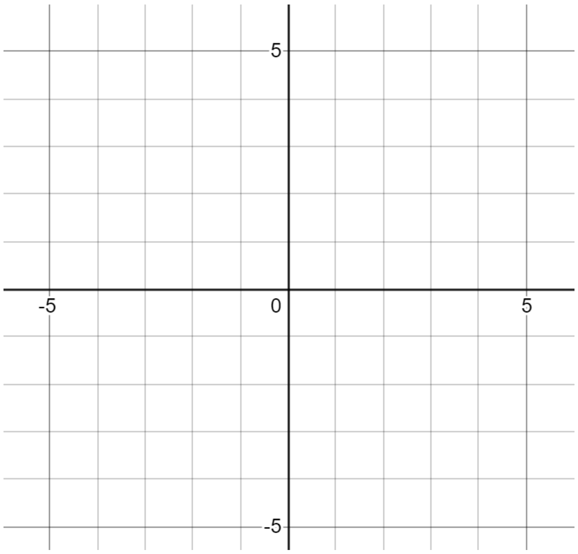


d) $line\ y = 2$

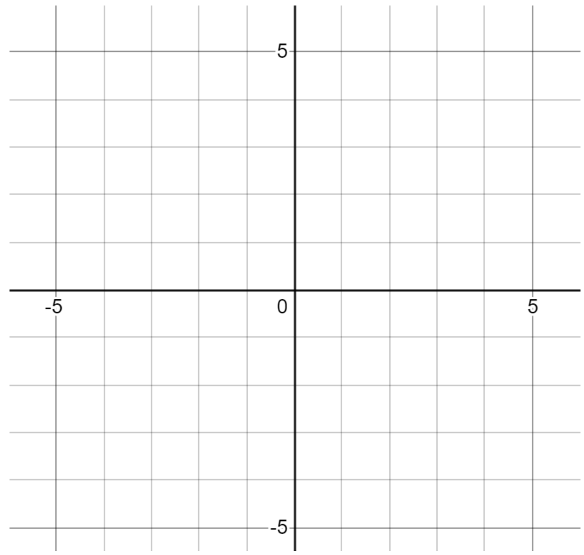


6. Graph the following functions without using Desmos, graph the basic form first, then graph the transformation and erase the original.

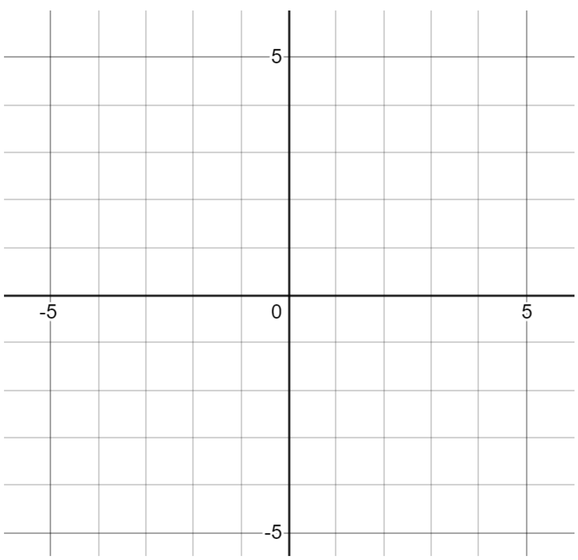
a) $f(x) = -(x-1)^2 + 3$



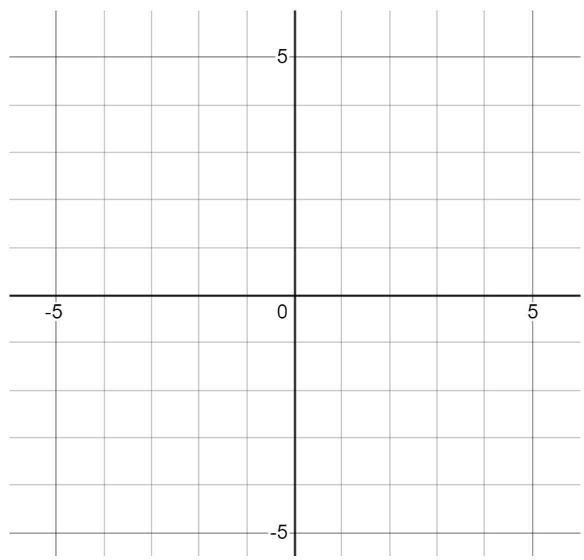
b) $f(x) = 3\sqrt{5-x} - 5$



c) $f(x) = -|1-x| + 3$

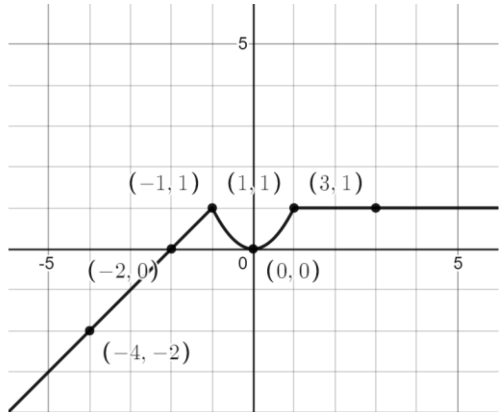


d) $f(x) = -\frac{1}{4}(x+2)^3 + 1$

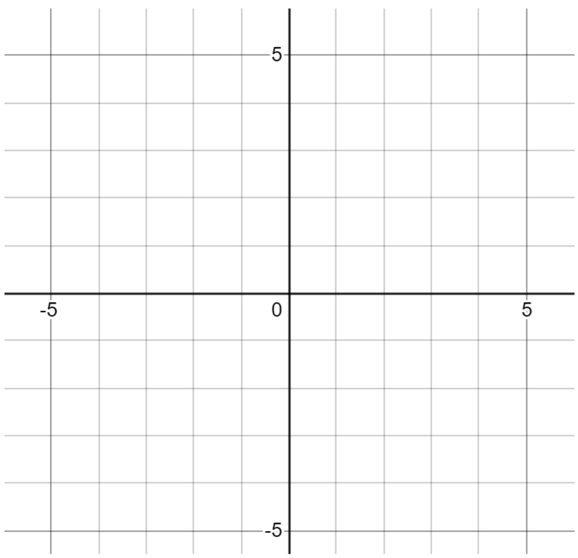


7. Given the graph of $y = f(x)$ below, sketch the graphs of the following:

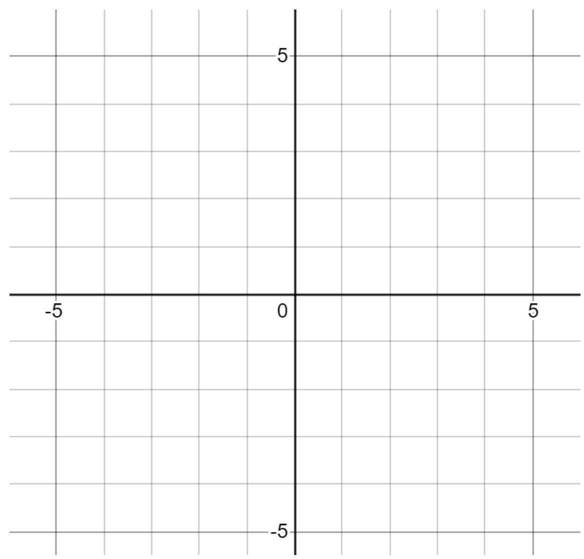
Use **Reference Points** to make this easier.



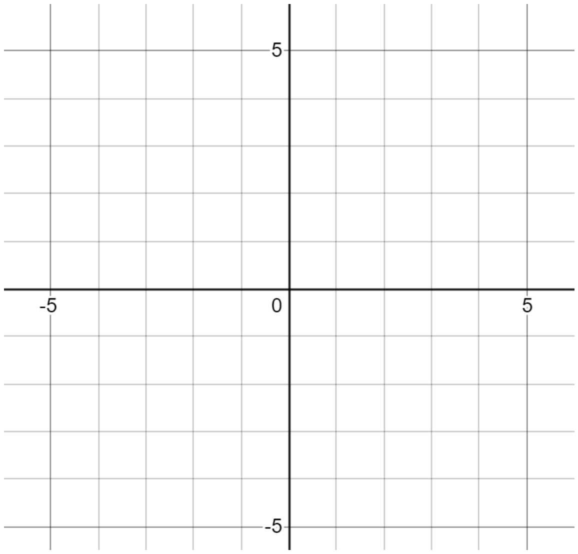
a) $y = f\left(\frac{1}{2}x\right) + 1$



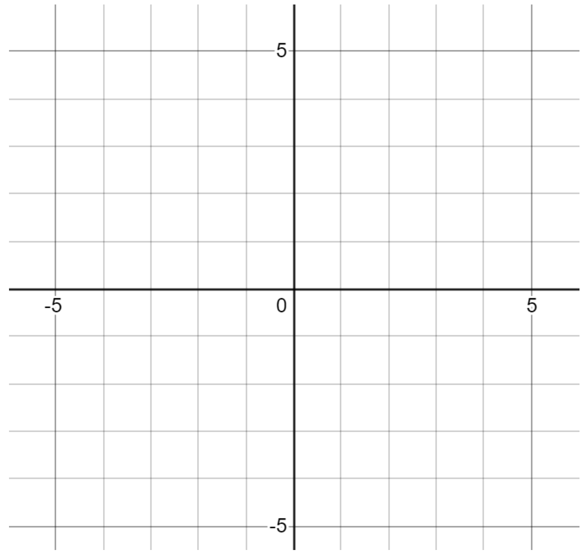
b) $y = -2f(x + 2) - 1$



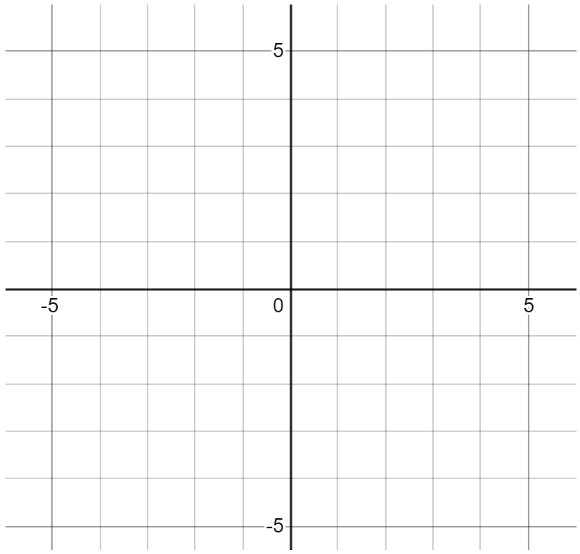
c) $y = 2f\left(\frac{1}{2}x - 1\right) + 1$



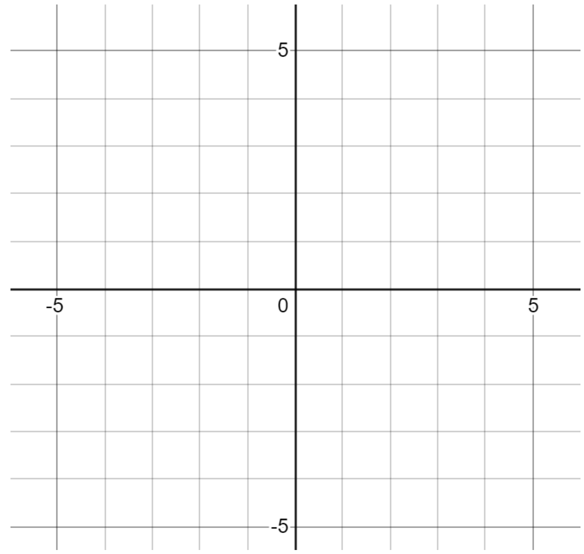
d) $y = 2f(1 - x) + 2$



e) $y = -f(2 - 2x) - 2$



f) $y = -2f\left(-\frac{1}{2}x - 1\right) + 1$



Answer Key for Number 1

1.

a) $(a+1, b+1)$	b) $(1-a, b)$	c) $(-a, -b)$
d) $a, b+1)$	e) $(-a, b)$	f) $(a, -b)$
g) $a-1, b)$	h) (b, a)	i) $(b, -a)$
j) $(b, a+1)$	k) $(b+1, a)$	l) $(-b, a+1)$
m) $(b, a+1)$	n) $(-b, 1-a)$	

See Website for Detailed Answer Key of the Remainder of the Questions

Extra Work Space