Section 2.5 – Inverse Functions

- We have **inverse operators**, like a zipper, they unzip what has been zipped
  - **Subtraction** is the **inverse of addition**
  - **Division** is the **inverse of multiplication**
  - **Square Rooting** is **the inverse of Squaring**

- **Functions have inverses** as well
  - Given two functions $f(x) and\ g(x)$, they are inverses of one another if one 'undoes' what the other 'does'.
  - We say then that $g(x) = f^{-1}(x)$ ← This reads $\boldsymbol{f\ inverse}$. It is NOT a negative exponent.

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

$$f\left(f^{-1}(x)\right) = x$$

$$and$$

$$f^{-1}(f(x)) = x$$

How to find the inverse of the Function $\boldsymbol{y = f(x)}$

1. Verify that f is a one-to-one function (otherwise the inverse is not a function)
2. Replace $f(x)\ with\ y$
3. Interchange $x\ and\ y$. (Change $x's\ to\ y's\ \ and\ \ y's\ to\ x's$)
4. Solve the new equation for y
5. Replace the new y with $f^{-1}(x)$

Example 1: Determine f^{-1} for $f(x) = 2x - 1$ and then verify the solution

Solution 1:

$f(x) = 2x - 1$	Function is one-to-one
$y = 2x - 1$	Replace $f(x)$ with y
$x = 2y - 1$	Interchange $x\ and\ y$
$x + 1 = 2y$	Solve for y
$\frac{x+1}{2} = y$	
$f^{-1}(x) = \frac{x+1}{2}$	Replace $y\ with\ f^{-1}(x)$

Check Solution:

$$f\left(f^{-1}(x)\right) = f\left(\frac{x+1}{2}\right)$$

$$= 2\left(\frac{x+1}{2}\right) - 1$$

$$= x + 1 - 1$$

$$= x$$

$$f^{-1}\left(f(x)\right) = f^{-1}(2x - 1)$$

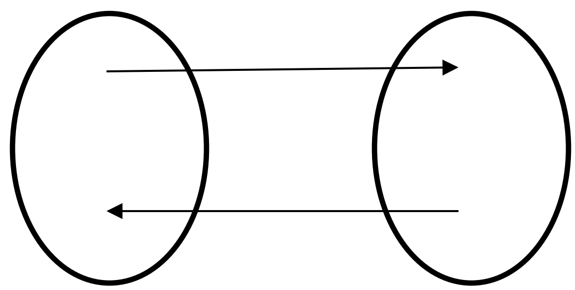
$$= \frac{2z - 1 + 1}{2}$$

$$= \frac{2x}{2}$$

$$= x$$

Therefore $f(x) and\ f^{-1}(x)$ are inverse functions.

- There is a Domain/Range relationship between inverse functions as well
- **The Domain of Function $f(x)$ is the Range of $f^{-1}(x)$**
- **The Range of Function $f(x)$ is the Domain of** $f^{-1}(x)$



Example 2: Determine g^{-1} for $g(x) = \sqrt{x-1}$

Solution 2:

$g(x) = \sqrt{x-1}$	Function is one-to-one
$y = \sqrt{x-1}$	Replace $g(x)$ with y
$x = \sqrt{y-1}$	Interchange x and y
$x^2 = y - 1$	Solve for y
$y = x^2 + 1$	
$g^{-1}(x) = x^2 + 1$	Replace y $with$ $g^{-1}(x)$

Check Solution:

$g(g^{-1}(x)) = g(x^2+1)$

$= \sqrt{x^2+1-1}$

$= \sqrt{x^2}$

$= |x|$ but $x \geq 0$

$= x$

$g^{-1}(g(x)) = g^{-1}(\sqrt{x-1})$

$= (\sqrt{x-1})^2 + 1$

$= x - 1 + 1$

$= x$

Therefore $g(x)$ and $g^{-1}(x)$ are inverse functions.

For $g(x) = \sqrt{x-1}$ and For $g^{-1}(x) = x^2 + 1$

Domain is: $x \geq 1$; Range is: $y \geq 0$ therefore Domain is: $x \geq 0$; Range is: $y \geq 1$

So, $g^{-1}(x) = x^2 + 1$, ***but it has a restricted Domain of*** $\boldsymbol{x \geq 0}$

This Domain restriction makes $g^{-1}(x)$ a one-to-one function thus given it an inverse in $g(x)$

Example 3: Determine h^{-1} of $h(x) = \dfrac{x}{2x-3}$

Solution 3:

$h(x) = \dfrac{x}{2x-3}$ Function is one-to-one

$y = \dfrac{x}{2x-3}$ Replace $h(x)$ with y

$x = \dfrac{y}{2y-3}$ Interchange $x\ and\ y$

$x(2y-3) = y$ Solve for y

$2xy - 3x = y$

$2xy - y = 3x$

$y(2x-1) = 3x$

$y = \dfrac{3x}{2x-1}$

$h^{-1}(x) = \dfrac{3x}{2x-1}$ Replace $y\ with\ h^{-1}(x)$

$h^{-1}(x) = \dfrac{3x}{2x-1},\ x \neq \dfrac{1}{2}$

Check Solution:

$$\boldsymbol{h\left(h^{-1}(x)\right)} = h\left(\frac{3x}{2x-1}\right)$$

$$= \frac{\frac{3x}{2x-1}}{2\left(\frac{3x}{2x-1}\right) - 3}$$

$$= \frac{\frac{3x}{2x-1}}{\left(\frac{6x}{2x-1}\right) - 3} \rightarrow \frac{\frac{3x}{2x-1}}{\left(\frac{6x}{2x-1}\right) - \frac{3(2x-1)}{(2x-1)}}$$

$$= \frac{\frac{3x}{2x-1}}{\left(\frac{6x-6x+3}{2x-1}\right)} = \frac{3x}{2x-1} \cdot \frac{2x-1}{x} = \frac{3x}{3} = x$$

$$\boldsymbol{h^{-1}\left(h(x)\right)} = h^{-1}\left(\frac{x}{2x-3}\right)$$

$$= \frac{3\left(\frac{x}{2x-3}\right)}{2\left(\frac{x}{2x-3}\right) - 1}$$

$$= \frac{\left(\frac{3x}{2x-3}\right)}{\left(\frac{2x}{2x-3}\right) - 1\frac{(2x-3)}{(2x-3)}}$$

$$= \frac{\left(\frac{3x}{2x-3}\right)}{\frac{2x-2x+3}{2x-3}} = \frac{\left(\frac{3x}{2x-3}\right)}{\frac{3}{2x-3}} = \frac{3x}{2x-3} \cdot \frac{2x-3}{3}$$

$$= \frac{3x}{3} = x$$

Therefore $h(x) and\ h(x)$ are inverse functions.

Example 4: Determine the inverse of $h(x) = x^2 + 2$

Solution 4: Since $h(x)$ is a parabola (quadratic, u-curved graph) then it is not one-to-one, so the inverse will not be a function but a Domain Restriction can change that, stay tuned.

$h(x) = x^2 + 2$	Function is not one-to-one
$y = x^2 + 2$	Replace $h(x)$ with y
$x = y^2 + 2$	Interchange $x\ and\ y$
$x - 2 = y^2$	Solve for y
$y = \pm\sqrt{x-2}$	Not a Function

Restrict Domain of $h(x) to\ x \geq 0$	**Restrict Domain of $h(x) to\ x \leq 0$**
Then $h^{-1}(x) = \sqrt{x+2}$ which is a Function	Then $h^{-1}(x) = -\sqrt{x+2}$ which is a Function

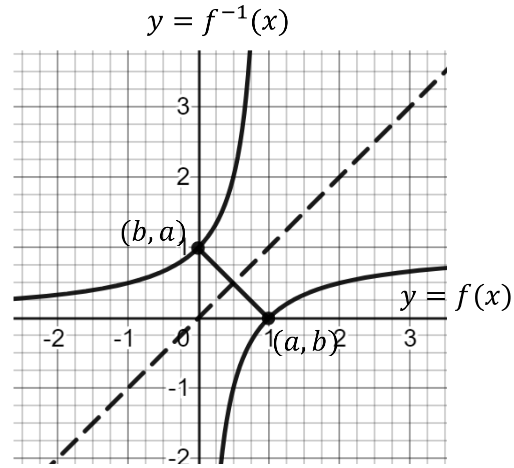
Check Solution:

$h\left(h^{-1}(x)\right) = h(\sqrt{x-2})$	$h\left(h^{-1}(x)\right) = h(-\sqrt{x-2})$
$= \left(\sqrt{x-2}\right)^2 + 2$	$= \left(-\sqrt{x-2}\right)^2 + 2$
$= x - 2 + 2$	$= x - 2 + 2$
$= x$	$= x$
$for\ x \geq 0$	$for\ x \geq 0$
$h^{-1}\left(h(x)\right) = h^{-1}(x^2 + 2)$	$h^{-1}\left(h(x)\right) = h^{-1}(x^2 + 2)$
$= \sqrt{x^2 + 2 - 2}$	$= -\sqrt{x^2 + 2 - 2}$
$= \sqrt{x^2}$	$= -\sqrt{x^2}$
$= x\ since\ x \geq 0$	$= x\ since\ x \leq 0$

Conclusion: for $x \geq 0, h(x)\ has\ in\ verse\ \sqrt{x-2}$

for $x \leq 0, h(x)\ has\ in\ verse\ -\sqrt{x-2}$

Graphs of Inverse Functions



- The graphs of f *and* f^{-1} are symmetric about the line $y = x$
- $f(x)$ is the **reflection** of $f^{-1}(x)$ **on the line $y = x$**
- The interesting part is threat what happens with this reflection, **the point (a, b) *on* $f(x)$**, becomes **the point (b, a)*on the graph of* $f^{-1}(x)$**

Example 5: Graph the inverse function of $g(x) = x^2, x \leq 0$

Solution 5: To find the inverse, we have to work through the following way:

$$y = x^2, \qquad x \leq 0, y \geq 0$$

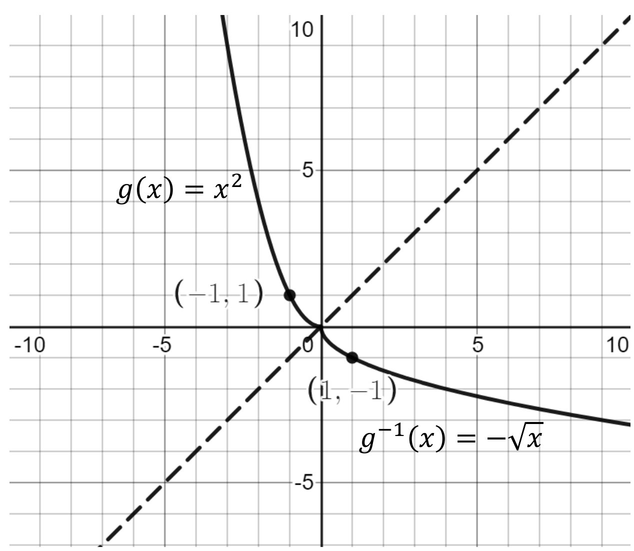
$$x = y^2, \qquad y \leq 0, x \geq 0$$

Swap, x's *and* y's in the Domain and Range too

$$y = \pm\sqrt{x}, \qquad y \leq 0, x \geq 0$$

y is negative, x is positive

So, $y = -\sqrt{x}$ and $g^{-1}(x) = -\sqrt{x},\ x \geq 0$



Example 6: $\boldsymbol{f(x) = 2x - 3}$

a) Determine $f^{-1}(x)$
b) Show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
c) Graph $f\ and\ f^{-1}$

Solution 6:

a)

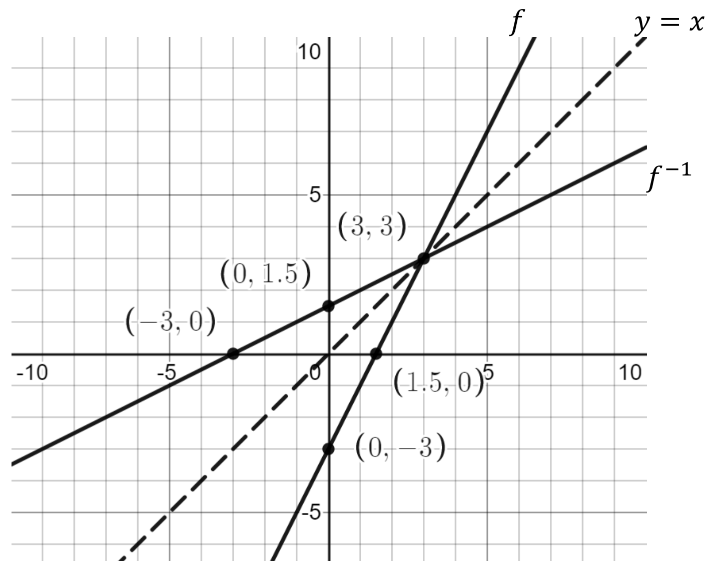
$$f(x) = 2x - 3$$
$$y = 2x - 3$$
$$x = 2y - 3$$
$$x + 3 = 2y$$
$$y = \frac{x+3}{2}$$

$$\boldsymbol{f^{-1}(x) = \frac{x+3}{2}}$$

b) $f(f^{-1}(x)) = f\left(\frac{x+3}{2}\right) = 2\left(\frac{x+3}{2}\right) - 3 = x + 3 - 3 = x$

$$f^{-1}(f(x)) = f^{-1}(2x-3) = \frac{(2x-3)+3}{2} = \frac{2x}{2} = x$$

c)



Transformation in Inverse Functions

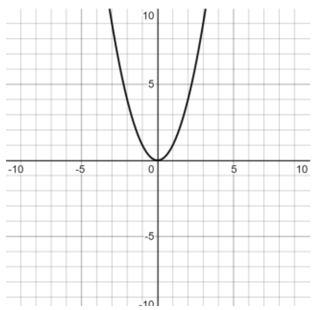
- The transformation process is exactly the same as before
- The only difference is that the **first step is to swap the $\boldsymbol{x, y - values}$**

For a point $\boldsymbol{(a, b)}$ in $\boldsymbol{y = f(x)}$

- $y = f^{-1}(x)$ will have a point (b, a)
- $y = f^{-1}(x-1)$ will have a point $(b+1, a)$
- $y = f^{-1}(x) + 1$ will have a point $(b, a+1)$
- $y = -2f^{-1}(3x)$ will have a point $(\frac{1}{3}b, -2a)$

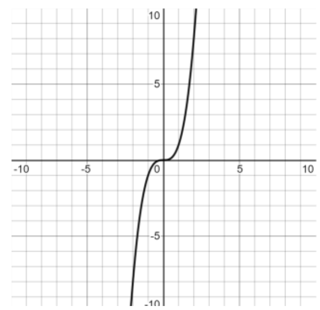
Section 2.5 – Practice Problems

1. The following are graphs of functions. Will they have inverse functions? Yes/No and Why?

a) 

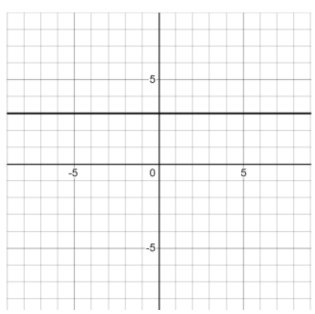
Yes or No

Why?

b) 

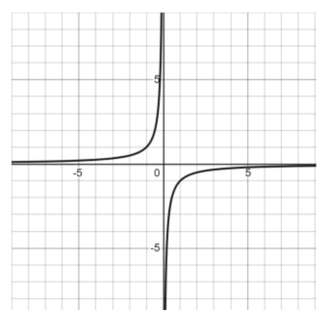
Yes or No

Why?

c) 

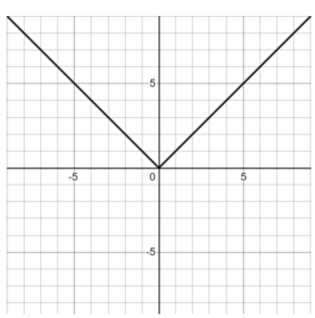
Yes or No

Why?

d) 

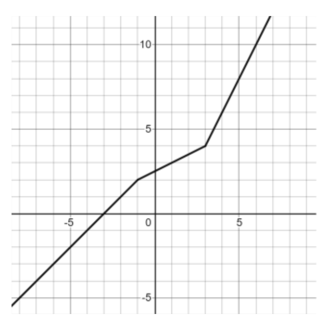
Yes or No

Why?

e) 

Yes or No

Why?

f) 

Yes or No

Why?

2. Determine whether the functions are inverses of each other by calculating $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x})$ *and* $(\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{x})$

a) $f(x) = \frac{3}{5}x,\ g(x) = \frac{5}{3}x$

b) $f(x) = x - 3,\ g(x) = x + 3$

c) $f(x) = 3 - 4x,\ g(x) = \frac{3 - x}{4}$

d) $f(x) = x^3 - 2,\ g(x) = \sqrt[3]{x + 2}$

e) $f(x) = \sqrt{x-1},\ g(x) = x^2 + 1$

f) $f(x) = \sqrt[4]{x}, x \geq 0,\ g(x) = x^4$

g) $f(x) = \dfrac{5x+3}{1-2x},\ g(x) = \dfrac{x-3}{2x+5}$

h) $f(x) = \sqrt[3]{x+1},\ g(x) = x^3 - 1$

3. Determine the restrictions on each of the following functions in order for its inverse to be a function

a) $f(x) = x^2$

b) $f(x) = x^2 + 2$

c) $f(x) = (x - 2)^2$

d) $f(x) = |x + 1| - 2$

4. Find the inverse of the following functions. State if the inverse is a function, a one-to-one function, or neither.

a) $f(x) = 2x - 3$

b) $f(x) = \sqrt{2x - 1}$

c) $f(x) = x^2 + 1$

d) $f(x) = \frac{1}{3x - 2}$

e) $f(x) = \frac{x}{1 - x}$

f) $f(x) = \frac{2x - 1}{3x + 2}$

5. Let $f(x) = 2x - 1, g(x) = \frac{1}{2}x + 3$, find $f^{-1}(x)$ and $g^{-1}(x)$, then determine

a) $(f^{-1} \circ g)(x)$

b) $(g^{-1} \circ f^{-1})(x)$

c) $(g \circ f^{-1})(x)$

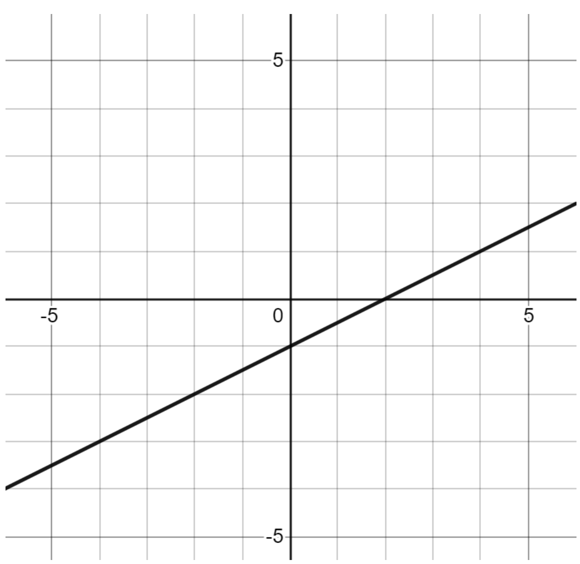
d) $(f \circ g^{-1})(x)$

e) $(f^{-1} \circ g^{-1})(x)$

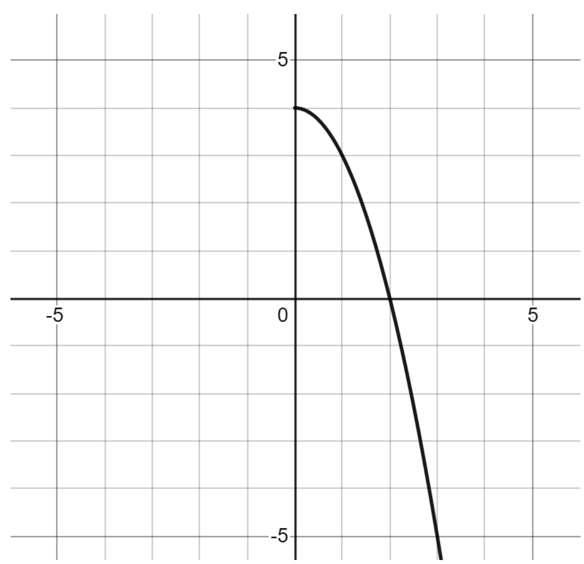
f) $(f \circ g)^{-1}(x)$

6. Given the graph of f, on the same grid draw the graph of the inverse of f.

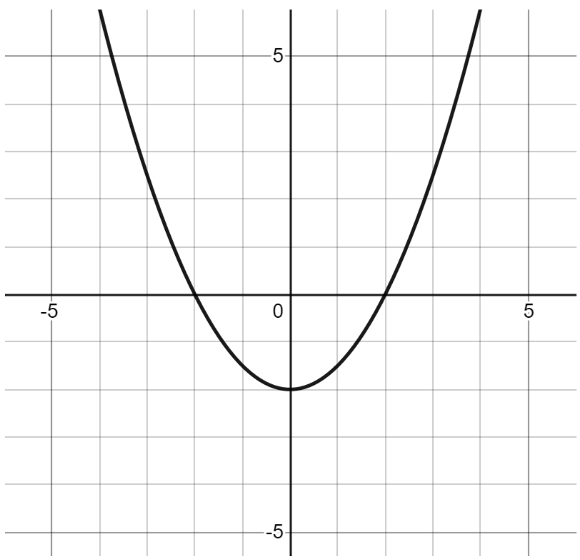
a)



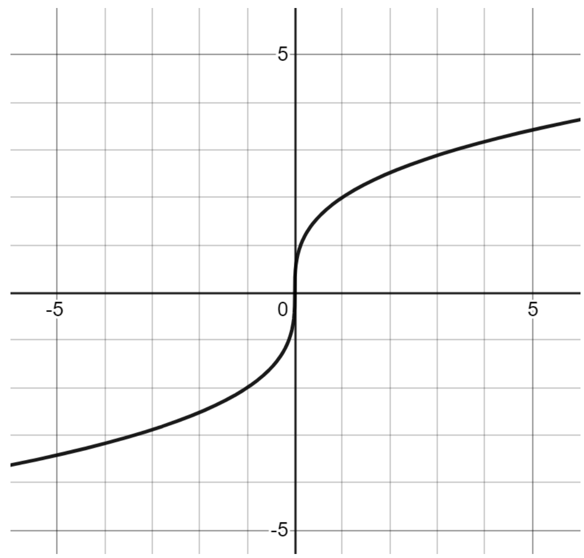
b)



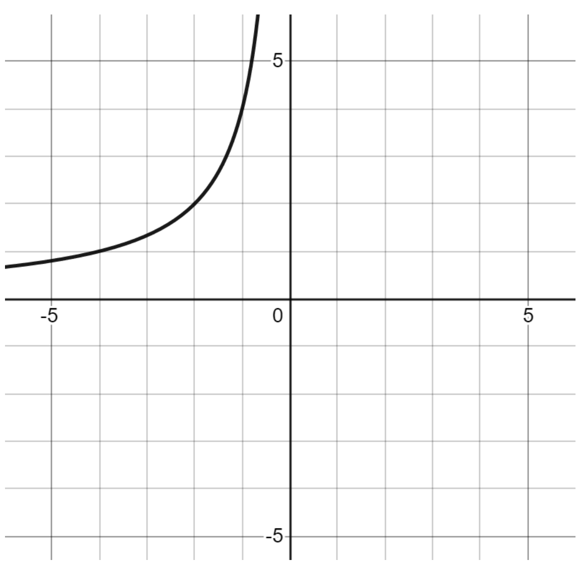
c)



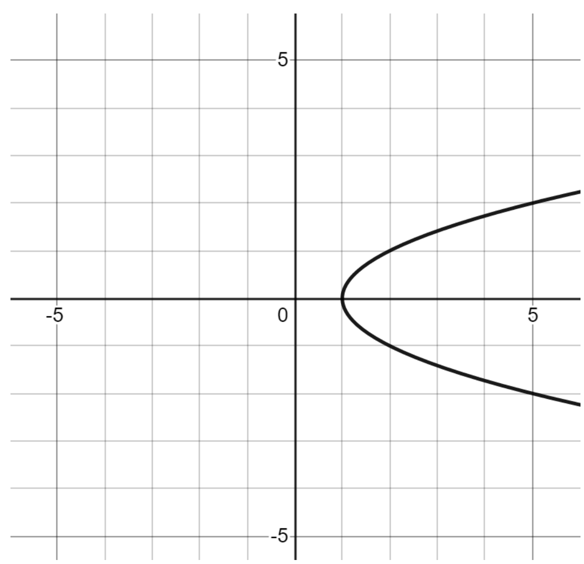
d)



e)



f)



7. If $(-1, 2)$ or (a, b) is a point of the graph of $y = f(x)$, what must be a point on the graph for the following?

a) $y = f^{-1}(x)$

b) $y = f^{-1}(x) - 1$

c) $y = f^{-1}(x + 2)$

d) $y = -f^{-1}(-x)$

e) $y = 1 - f^{-1}(-x)$

f) $y = f^{-1}(x + 1)$

8. Use Desmos to graph the following functions and their inverses. State if the inverse is a function, a one-to-one function, or neither.

a) $f(x) = 2x - 1$

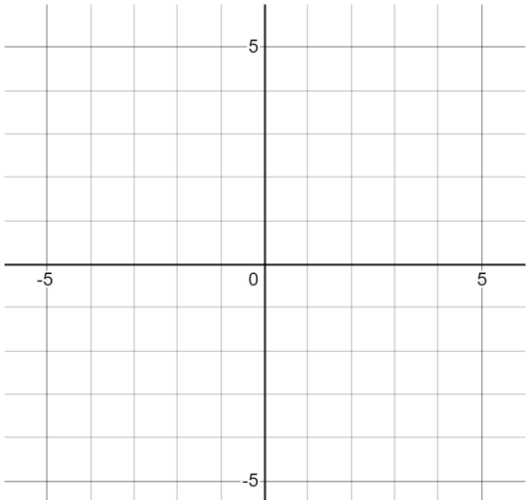
b) $f(x) = x^2 + 1$

c) $f(x) = x^3 - 1$

d) $f(x) = \sqrt{x^2 - 4}$

9. The function $f(x) = a(-x^3 - x + 2)$ has an inverse function such that $f^{-1}(6) = -2$. Find a.

10. If the graph of f contains points in Quadrant I and II, the graph of f^{-1} must contain points in which Quadrant(s)? (Use the grid provided to help visualize)



11. The formulas for Fahrenheit and Celsius temperatures are:

$$F = \frac{9}{5}C + 32 \quad and \quad C = \frac{5}{9}(F - 32)$$

Show that these two functions are inverses of each other.

12. Show that for the one-to-one function $f(x) = 2x + 1$ and $g(x) = \frac{1}{4}x - 3$, that:

$$(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$$

See Website Copy for Detailed Answer Key

Extra Work Space