Section 2.4 – Transformations of Graphs

- Transformations is when we change the basic graph of a function in 2-dimensional space
- In this section, we will look at:
    - **Translations** – vertical and horizontal shifts
    - **Compression and Expansion** – stretch and squeeze
    - **Reflections** – in both the $x\ and\ y\ axes$

- If we consider a basic function: $\boldsymbol{y = f(x)}$

This can seem a little daunting, so we will look at it piecewise.

Transformations can give us shifts represented by: $\boldsymbol{y = af[b(x - c)] + d}$

1. **Translations**, or shifts, are additions or subtractions **represented by $\boldsymbol{c\ and\ d}$**
2. **Expansions**, or compressions, are multiplications **shown by $\boldsymbol{a\ and\ b}$**
3. **Reflections** happen **when $\boldsymbol{a\ or\ b}$ are negative**

- **Constants $\boldsymbol{a\ and\ d}$**, which are **"outside of the function"**, affect the $\boldsymbol{y - values}$ of the ordered pairs
- **Constants $\boldsymbol{b\ and\ c}$**, which are **"inside the function"**, affect the $\boldsymbol{x - values}$ of the ordered pairs

This is a big deal and can help us make this process as simple as possible!!

- Let's look at these various transformations separately.

Translations

A translation is when the graph **is shifted in the left or right ($\boldsymbol{x\ direction}$)** or the **up and down** $(\boldsymbol{y\ direction})$, without changing the shape of the original graph

a) Vertical Translations ($\boldsymbol{y\ direction}$), $\boldsymbol{d > 0}$

If $d > 0$, for the graph of $y = f(x)$, the graph of:

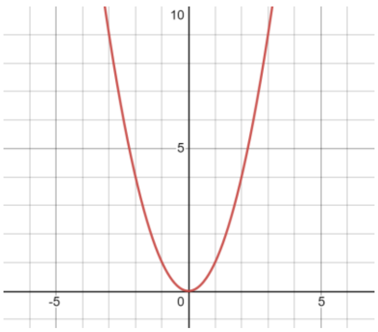
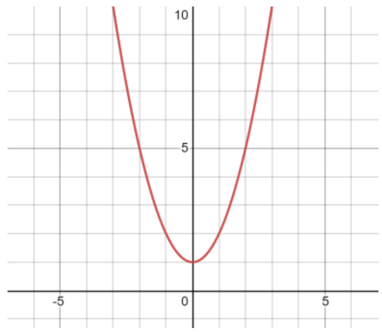
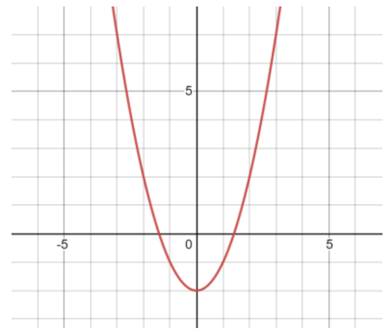
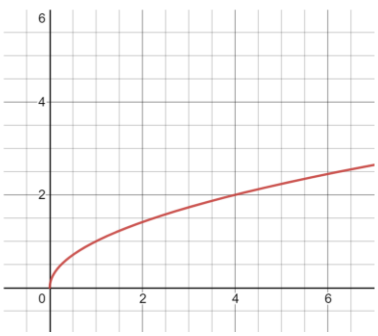
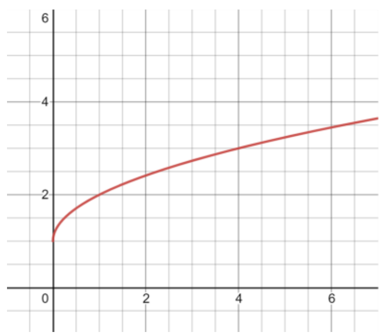
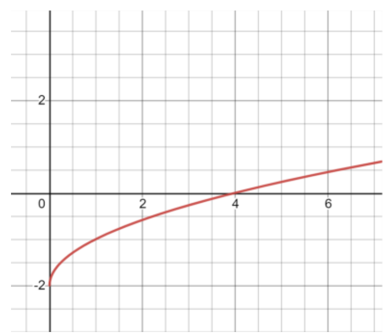
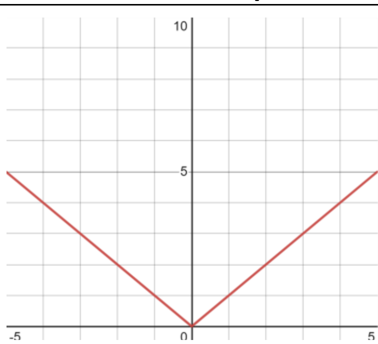
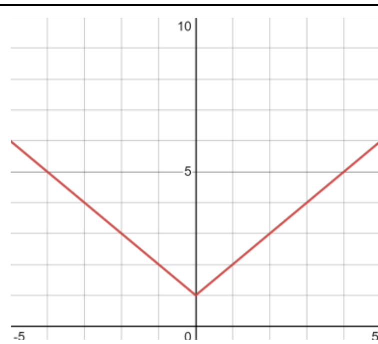
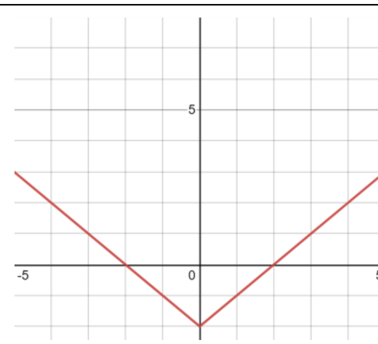
$\boldsymbol{y = f(x) + d}$ is **shifted up "$\boldsymbol{d}$" units**

$\boldsymbol{y = f(x) - d}$ is **shifted down "$\boldsymbol{d}$" units**

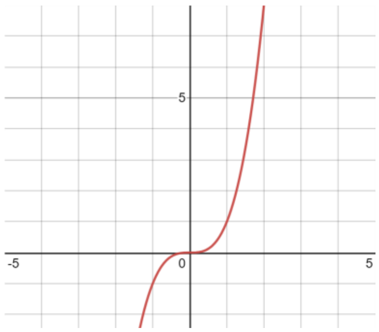
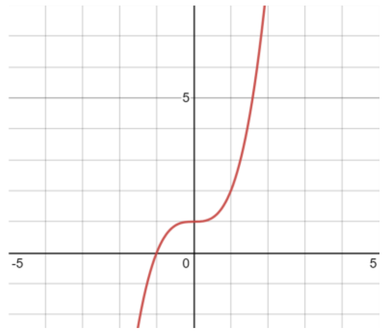
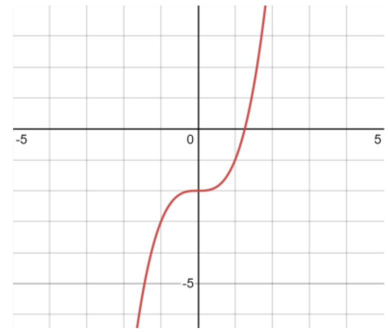
Vertical Translations are quite **intuitive**, they literally **move up or down** depending of the **sign and number of the $\boldsymbol{d\ value}$**

See the following graphs as examples of vertical translations

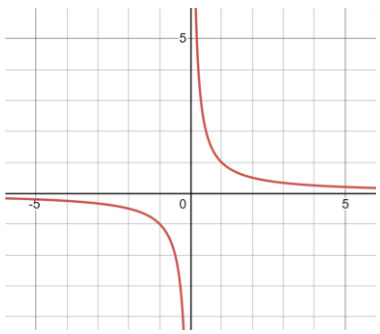
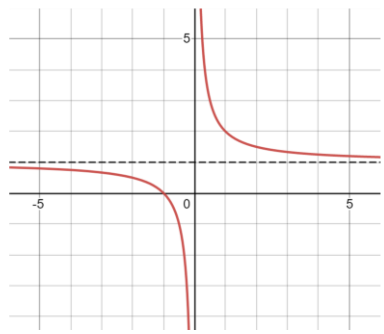
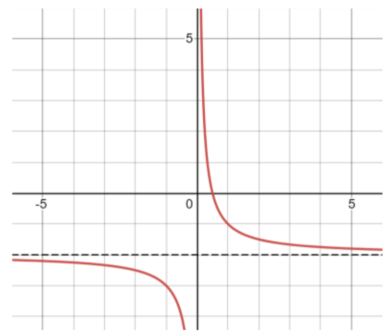
Example 1:

Quadratic Graphs		
 $y = x^2$ $y = f(x)$	 $y = x^2 + 1$ $y = f(x) + 1$	 $y = x^2 - 2$ $y = f(x) - 2$
Square Root Graphs		
 $y = \sqrt{x}$ $y = f(x)$	 $y = \sqrt{x} + 1$ $y = f(x) + 1$	 $y = \sqrt{x} - 2$ $y = f(x) - 2$
Absolute Value Graphs		
 $y = \|x\|$ $y = f(x)$	 $y = \|x\| + 1$ $y = f(x) + 1$	 $y = \|x\| - 2$ $y = f(x) - 2$

Cubic Graphs

 $y = x^3$ $y = f(x)$	 $y = x^3 + 1$ $y = f(x) + 1$	 $y = x^3 - 2$ $y = f(x) - 2$

Reciprocal Graphs

 $y = \frac{1}{x}$ $y = f(x)$	 $y = \frac{1}{x} + 1$ $y = f(x)$	 $y = \frac{1}{x} - 2$ $y = f(x)$

b) Horizontal Translations ($\boldsymbol{x\ direction}$), $\boldsymbol{c > 0}$

If $c > 0$, for the graph of $y = f(x)$, the graph of:

$\boldsymbol{y = f(x + c)}$ is **shifted left "$\boldsymbol{c}$" units**

$\boldsymbol{y = f(x - c)}$ is **shifted right "$\boldsymbol{c}$" units**

> **Horizontal Translations** are not **intuitive**, they **move the opposite direction** of the **sign of the $\boldsymbol{c\ value}$**

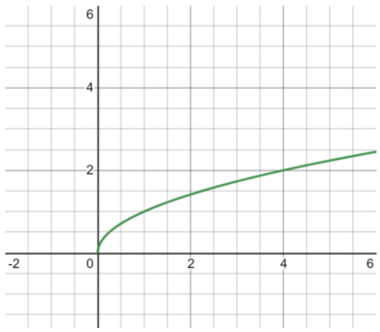
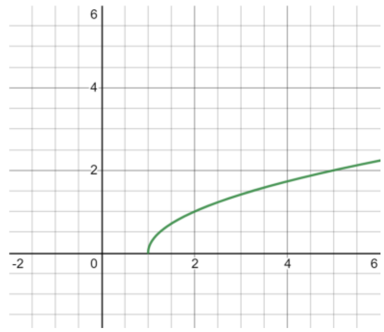
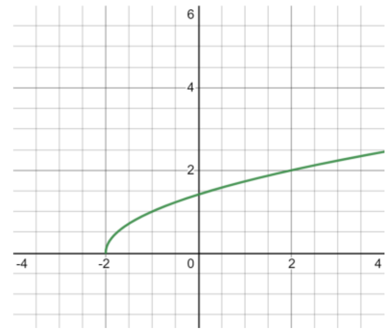
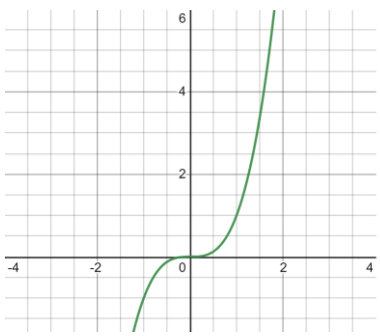
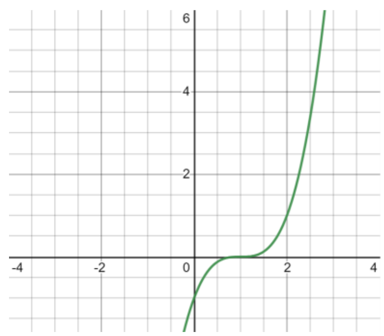
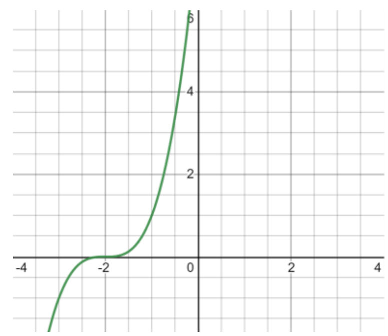
> I like to think to consider "what value of x makes the inside zero". That value is where you move on the $x - axis$.

$$y = f(x - 3) \qquad or \qquad y = f(x + 2)$$

> Moves right 3, or $x = 3$ makes $x - 3 = 0$

> Moves left 2, or $x = -2$ makes $x + 2 = 0$

Example 2:

Square Root Graphs		
 $y = \sqrt{x}$ $y = f(x)$	 $y = \sqrt{x-1}$ $y = f(x-1)$	 $y = \sqrt{x+2}$ $y = f(x+2)$
Cubic Graphs		
 $y = x^3$ $y = f(x)$	 $y = (x-1)^3$ $y = f(x-1)$	 $y = (x+2)^3$ $y = f(x+2)$

Summary

Vertical and Horizontal Translations of $y = f(x)$ with point (x, y) If $c, d > 0$:	
1. Vertical translation of d units *upward*	$h(x) = f(x) + d, \ (x, \boldsymbol{y + d})$
2. Vertical translation of d units *downward*	$h(x) = f(x) - d, \ (x, \boldsymbol{y - d})$
3. Horizontal translation of c units *to the right*	$h(x) = f(x - c), \ (\boldsymbol{x + c}, y)$
4. Horizontal translation of c units *to the left*	$h(x) = f(x + c), \ (\boldsymbol{x - c}, y)$

Example 3: Write the equation of the function $f(x) = \sqrt{x}$ after a transformation $\boldsymbol{4\ units\ right\ and\ 3\ units\ down}$

Solution 3: $g(x) = \sqrt{x-4} - 3$ ← Three units down

↑ To the right means $x - 4$

Example 4: What transformations have occurred to change $\boldsymbol{y = f(x)}$ **into** $\boldsymbol{y = f(x-2) + 4}$?

Solution 4: Horizontal translation: $2\ units\ right$ Vertical Translation: $4\ units\ up$

Example 5: If $(2, 2)$ is in $y = f(x)$, which point is on $y = f(x+3) - 2$ ← This moves the $\boldsymbol{y - coordinate}$ **down 2 units**

↑ This moves the $\boldsymbol{x - coordinate}$ **left 3 units**

Solution 5: $(x - 3, y - 2)$

$(2 - 3, 2 - 2) \quad \rightarrow \quad \boldsymbol{(-1, 0)}$

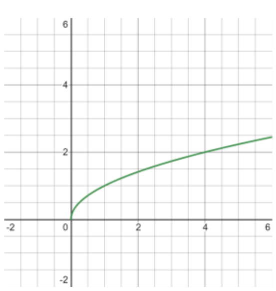
Reflections

The next type of transformation is a reflection. We are going to talk about reflecting over the $x - axis$ and $y - axis$ only.

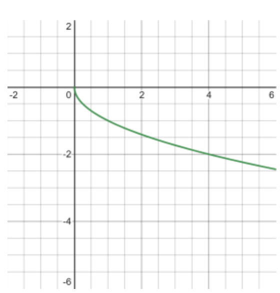
- Consider reflecting over the $x - axis$, all $y - values$ change their signs.
- Consider reflecting over the $y - axis$, all $x - values$ change their signs.

For the graph of $y = f(x)$, the graph of:

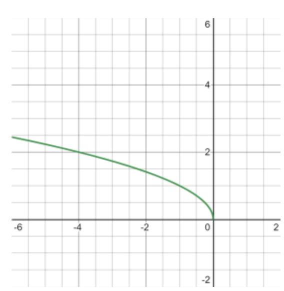
- $\boldsymbol{y = -f(x)}$ is a **refection of the** $\boldsymbol{y - values}$, a **reflection in the** $\boldsymbol{x - axis}$
- $\boldsymbol{y = f(-x)}$ is a **refection of the** $\boldsymbol{x - values}$**,** a **reflection in the** $\boldsymbol{y - axis}$
- $\boldsymbol{y = -f(-x)}$ is a **refection of the** $\boldsymbol{x\ and\ y - values}$, a reflection in the $\boldsymbol{x\ and\ y - axis}$



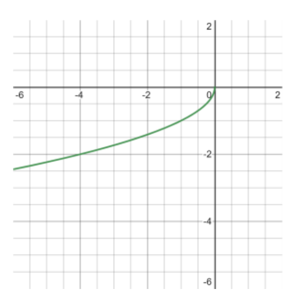
$y = \sqrt{x}$
$y = f(x)$



$y = -\sqrt{x}$
$y = -f(x)$



$y = \sqrt{-x}$
$y = f(-x)$



$y = -\sqrt{-x}$
$y = -f(-x)$

Summary

Reflections of $\boldsymbol{y = f(x)}$ with point $\boldsymbol{(x, y)}$ in the two Axes	
1. Reflection in the $x - axis$	$h(x) = -f(x),\ \ (x, -\boldsymbol{y})$
2. Reflection in the $y - axis$	$h(x) = f(-x),\ \ (-\boldsymbol{x}, y)$
3. Reflection in both $axes$	$h(x) = -f(-x),\ \ (-\boldsymbol{x}, -\boldsymbol{y})$

Example 6: Write the equation of the function $f(x) = x^2 + x$ if it is reflected in the:

a) $x - axis$
b) $y - axis$

Solution 6:

a) $f(x) \rightarrow -f(x)$ so $x^2 + x \rightarrow -(x^2 + x) = -x^2 - x$

b) $f(x) \rightarrow f(-x)$ so $x^2 + x \rightarrow (-x)^2 + (-x) = x^2 - x$

Example 7: What transformations have occurred to change $\boldsymbol{y = x^2 + 2x}$ **into** $\boldsymbol{y = -(x^2 + 2x)}$?

Solution 7: Since the entire original function is inside the brackets, the negative on the outside. It is a **reflection of the $\boldsymbol{y - values\ (the\ x - axis)}$**.

Example 8: If $(3, 2)$ is in $y = f(x)$, which point is on:

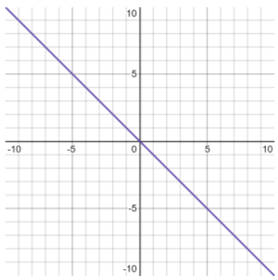
a) $y = -f(x)$
b) $y = f(-x)$
c) $y = -f(-x)$

Solution 8:

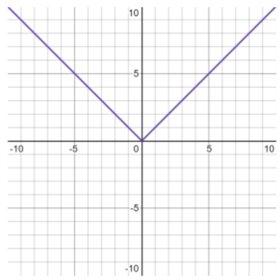
a) Sign change in $\boldsymbol{y - values}$**:** $\boldsymbol{(3, -2)}$
b) Sign change in $\boldsymbol{x - values}$**:** $\boldsymbol{(-3, 2)}$
c) Sign change in $\boldsymbol{x\ and\ y - values}$**:** $\boldsymbol{(-3, -2)}$

Absolute Value Function

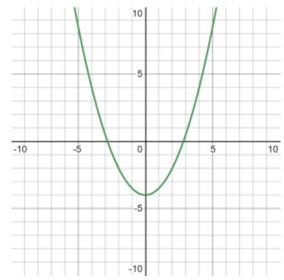
- The Domain $(x - values)$ of an absolute value function $y = |f(x)|$ is the same as the original function $f(x)$
- But since absolute value cannot to negative
- The Range $(y - values)$ of an absolute value function $y = |f(x)|$only has positive values $y = f(x) \geq 0$



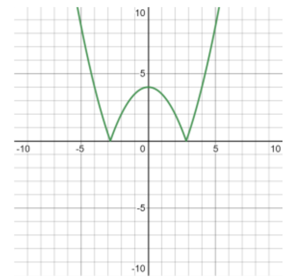
$y = f(x)$



$y = |f(x)|$



$y = g(x)$



$y = |g(x)|$

Reciprocal Function

- If $f(x)$ then the **reciprocal function** has the form: $\mathbf{\frac{1}{f(x)}}$

- This means all the $\boldsymbol{y - values\ (outputs)}$ **become reciprocals**
- I will not cover this is too much detail here (see the video on Reciprocal Functions), but see the example below.

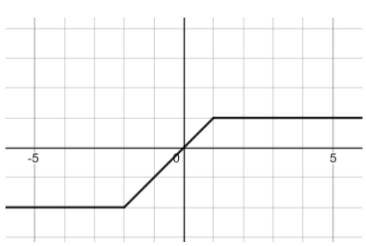
Example 9: If $y = f(x)$ has the coordinate point $(-2,4)$, what point is on $\frac{1}{f(x)}$

Solution 9: The Domain $(x - values)$ do not change but the Range $(y - values)$ become reciprocals of their original graphs

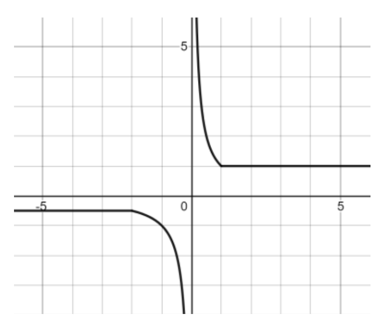
So $(-2, 4) \quad \rightarrow \quad \left(\mathbf{-2, \frac{1}{4}}\right)$

Example 10: Given the graph of $f(x)$ below, graph the reciprocal function

Solution 10:



- All outputs become reciprocals
- Where $y = 0$ we end up with vertical asymptotes
- Be considerate of the infinitely increasing and decreasing limits



Compression and Expansion of Graphs

- Vertical and horizontal shifts leave the shape of the graph the same
- Compressions and Expansions graph a shape change, either a squeeze of a stretch
- There are helpful markers to determine whether or not it is a Vertical or Horizontal stretch

a) Vertical Compression and Expansion

For the graph of $y = f(x)$, the graph of:

$\boldsymbol{y = a \cdot f(x)}$ is a **Vertical Expansion if $\boldsymbol{a > 1}$ (Expansion by a factor of $\boldsymbol{a}$)**

$\boldsymbol{y = a \cdot f(x)}$ is a **Vertical Compression if $\boldsymbol{0 < a < 1}$ (Compression by a factor of $\boldsymbol{a}$, where a is a proper fraction)**

For the graph of $y = f(x)$, the graph of:

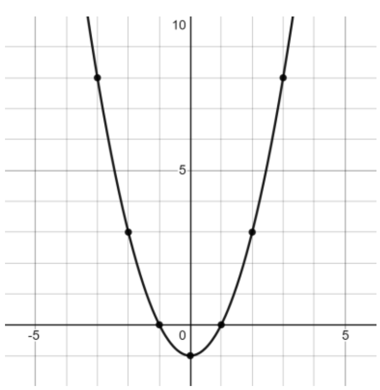
$y = 2f(x)$ is a **Vertical Expansion** by a **factor of 2**

$y = \frac{1}{3}f(x)$ is a **Vertical Compression** by a factor of $\frac{1}{3}$

Vertical Expansions and Compressions <u>keep the $\boldsymbol{x - intercepts}$</u> of the original function!

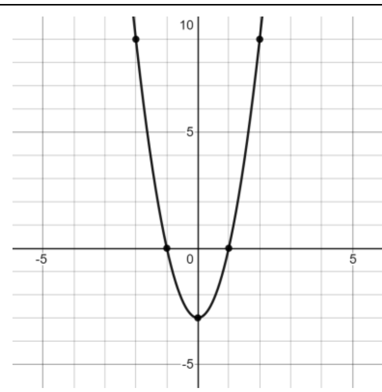
<u>Example 11:</u>

Quadratic Graphs



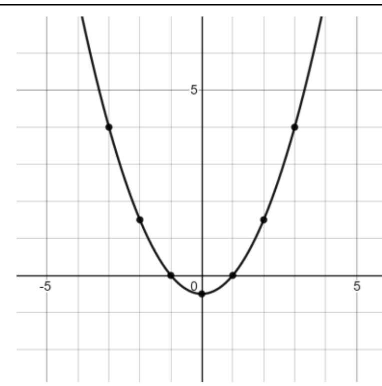
$y = x^2 - 1$
$y = f(x)$

Basic Graph



$y = 3(x^2 - 1)$
$y = 3f(x)$

Vertical Expansion by a Factor of 3



$y = \frac{1}{2}(x^2 - 1)$
$y = \frac{1}{2}f(x)$

Vertical Compression by a Factor of $\frac{1}{2}$

You see the $\boldsymbol{x - intercepts}$ did not change, but the shape of the graph was altered

b) Horizontal Compressions and Expansion

For the graph of $y = f(x)$, the graph of:

$\boldsymbol{y = f(bx)}$ is a **Horizontal Compression if $\boldsymbol{b > 1}$ (by a factor of $\boldsymbol{\frac{1}{b}}$)**

$\boldsymbol{y = f(bx)}$ is a **Horizontal Expansion if $\boldsymbol{0 < b < 1}$ (by a factor of $\boldsymbol{\frac{1}{b}}$ where $\boldsymbol{b}$ is a proper fraction)**

Horizontal Expansions and Compressions keep **<u>the $\boldsymbol{y-intercept}$</u>** of the original function!

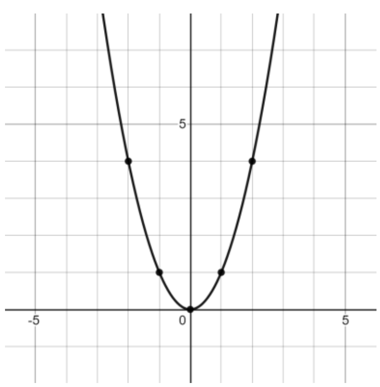
For the graph of $y = f(x)$, the graph of:

$y = f(\boldsymbol{2x})$ is a **Horizontal Compression** by a **factor of $\boldsymbol{\frac{1}{2}}$**

$y = f(\boldsymbol{\frac{1}{3}x})$ is a **Horizontal Expansion** by **a factor of $\boldsymbol{3}$**

<u>Example 11:</u>

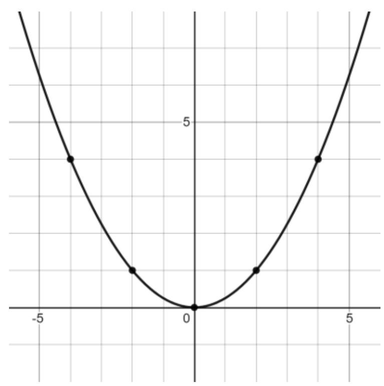
Quadratic Graphs



$y = x^2$

$y = f(x)$

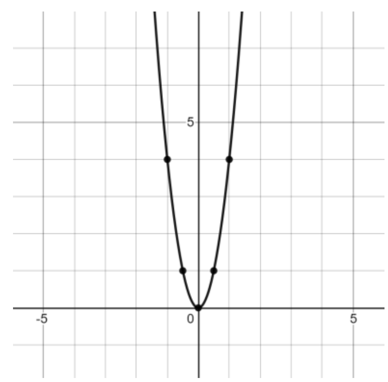
$Basic\ Graph$



$y = \left(\frac{1}{2}x\right)^2$

$y = f\left(\frac{1}{2}x\right)$

$Horizontal\ Expansion\ by\ a\ Factor\ of\ 2$



$y = (2x)^2$

$y = f(2x)$

$Horizontal\ Compression\ by\ a\ Factor\ of\ \frac{1}{2}$

You see the $\boldsymbol{y-intercepts}$ did not change, but the shape of the graph was altered

Summary

Vertical and Horizontal Compressions and Expansions of $\boldsymbol{y = f(x)}$ **with point** $\boldsymbol{(x, y)}$ If $a > 1, b > 1$:	
1. Vertical expansion by a factor of a	$h(x) = af(x),\ (x, \boldsymbol{ay})$
2. Horizontal compressions by a factor of $\frac{1}{b}$	$h(x) = f(bx),\ (\boldsymbol{\frac{1}{b}x}, y)$
If $0 < a < 1, 0 < b < 1$:	
3. Vertical expansion by a factor of a (a is a proper fraction)	$h(x) = af(x),\ (x, \boldsymbol{ay})$
4. Horizontal compressions by a factor of $\frac{1}{b}$ (b is the reciprocal of a proper fraction)	$h(x) = f(bx),\ (\boldsymbol{bx}, y)$

Example 12: Write an equation for the function $y = \sqrt{x}$, with a

a) Vertical Expansion by a factor of 2

b) Vertical Compression by a factor of $\frac{1}{2}$

c) Horizontal Expansion by a factor of 2

d) Horizontal Compression by a factor of $\frac{1}{2}$

Solution 12:

a) $y = 2\sqrt{x}$ b) $y = \frac{1}{2}\sqrt{x}$ c) $y = \sqrt{\frac{1}{2}x}$ d) $y = \sqrt{2x}$

Example 13: What transformation has happened to $y = f(x)$ to produce $y = 3f(\frac{1}{4}x)$?

Solution 13:

- ✓ **Vertical expansion** by a **factor of 3**
- ✓ **Horizontal expansion** by a **factor of** $\boldsymbol{\frac{1}{\frac{1}{4}} \rightarrow 4}$

Example 14: If $(3, 1)$ is on $y = f(x)$, what point is on $y = 2f(4x)$?

Solution 14:

$$(x, y) \rightarrow \left(\frac{1}{4}x, 2y\right) \rightarrow \left(\frac{1}{4}(3), 2(1)\right) \rightarrow \boldsymbol{\left(\frac{3}{4}, 2\right)}$$

Section 2.4 – Practice Problems

1. Write an equation for the function that is described by the given characteristics.

a) The shape $f(x) = x^2$, moved 4 $units$ to the left and 5 $units$ downward.

b) The shape $f(x) = x^2$, moved 2 $units$ to the right, reflected in the $x - axis$, and moved 3 $units$ upward.

c) The shape $f(x) = x^3$, moved 2 $units$ to the right and 3 $units$ downward.

d) The shape $f(x) = x^3$, moved 1 $unit$ downward and reflected in the $y - axis$.

e) The shape $f(x) = |x|$, moved 6 $units$ upward and 3 $units$ to the left.

f) The shape $f(x) = |x|$, moved 3 $units$ to the left and reflected in the $x - axis$

g) The shape $f(x) = \sqrt{x}$, moved 7 $units$ to the right and reflected in the $x - axis$

h) The shape $f(x) = \sqrt{x}$, moved 4 $units$ upward and reflected in the $y - axis$

2. If $(-3, 1)$ or (a, b) is a point on the graph of $y = f(x)$, what must be a point on the graph of the following?

a) $y = f(x+2)$

b) $y = f(x) + 2$

c) $y = f(x-2) - 2$

d) $y = -f(x)$

e) $y = f(-x)$

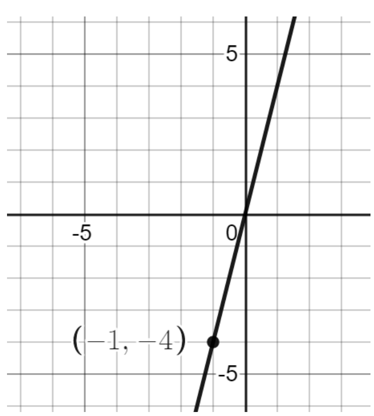
f) $y = -f(-x)$

g) $y = f(-x) - 2$

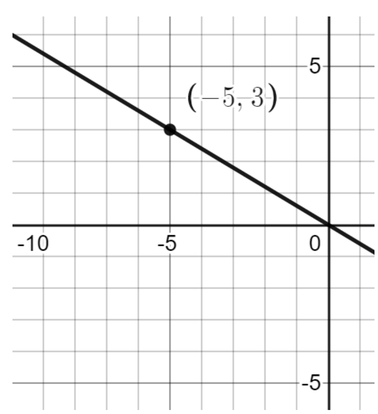
h) $y = -f(x+2)$

3. Use the graph of $f(x) = x$ to write an equation for each function whose graph is shown. Each transformation includes only reflections or expansions/compressions.

a)

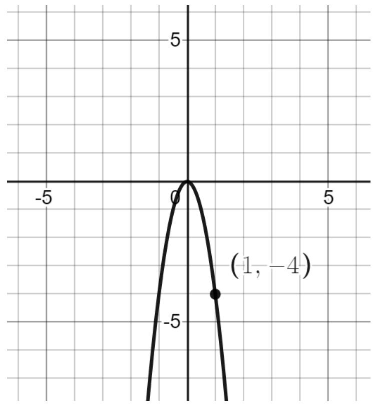


b)

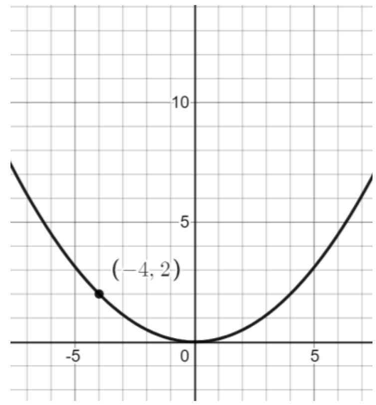


4. Use the graph of $f(x) = x^2$ to write an equation for each function whose graph is shown. Each transformation includes only reflections or expansions/compressions.

a)

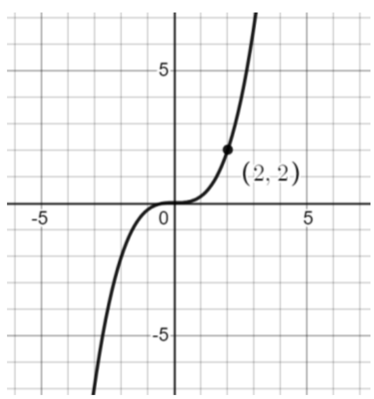


b)

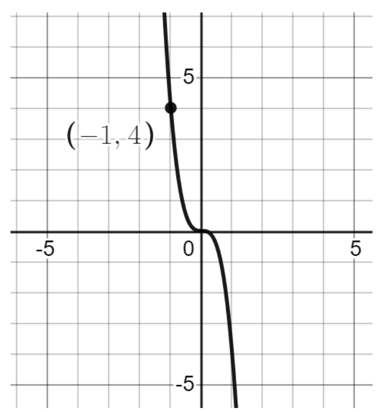


5. Use the graph of $f(x) = x^3$ to write an equation for each function whose graph is shown. Each transformation includes only reflections or expansions/compressions.

a)

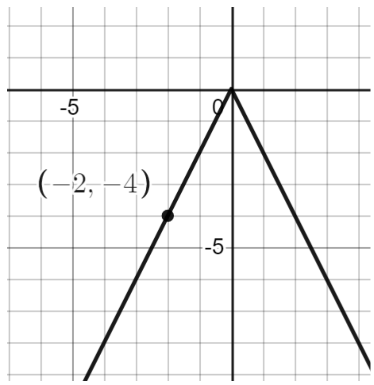


b)

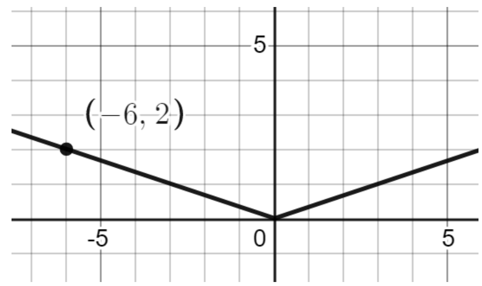


6. Use the graph of $f(x) = |x|$ to write an equation for each function whose graph is shown. Each transformation includes only reflections or expansions/compressions.

a)

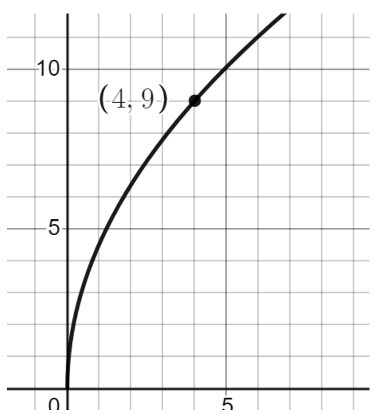


b)

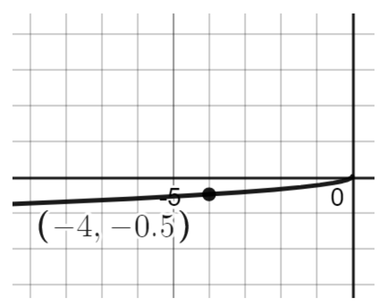


7. Use the graph of $f(x) = \sqrt{x}$ to write an equation for each function whose graph is shown. Each transformation includes only reflections or expansions/compressions.

a)

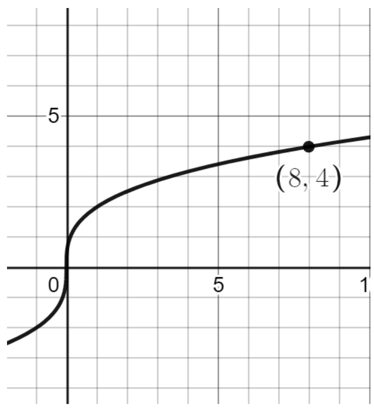


b)

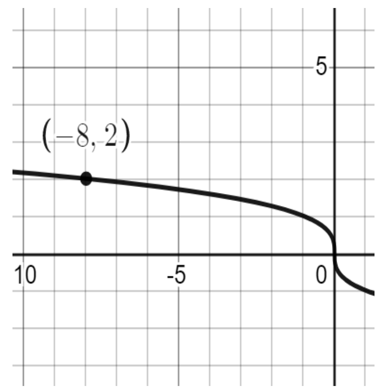


8. Use the graph of $f(x) = x^{\frac{1}{3}}$ to write an equation for each function whose graph is shown. Each transformation includes only reflections or expansions/compressions.

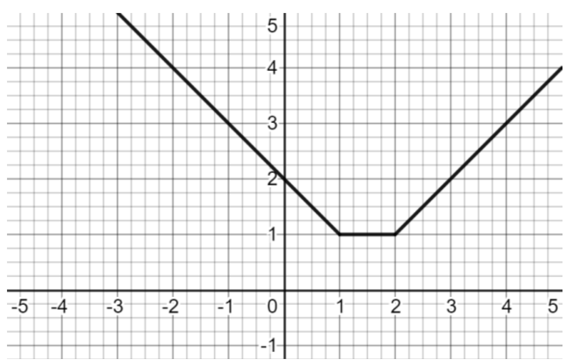
a)



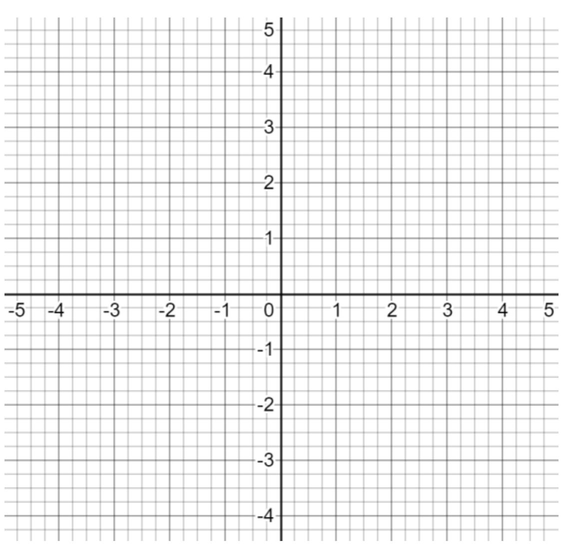
b)



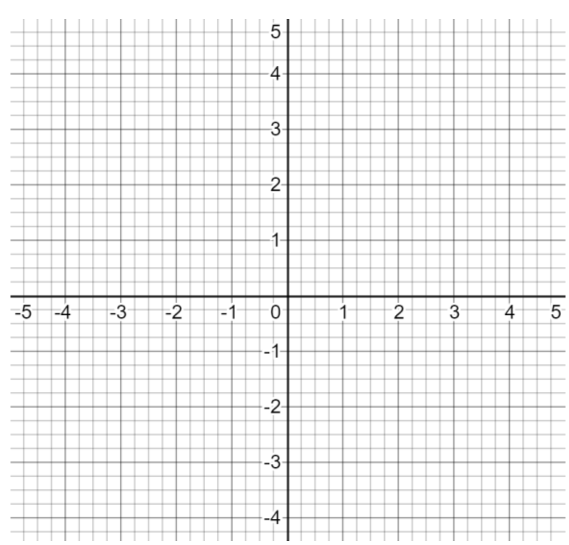
9. Given the graph of $f(x)$ below, sketch the graphs of the following:



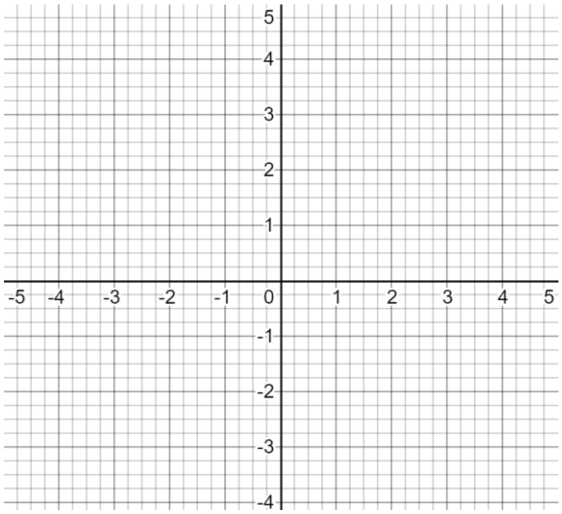
a) $y = -f(x)$



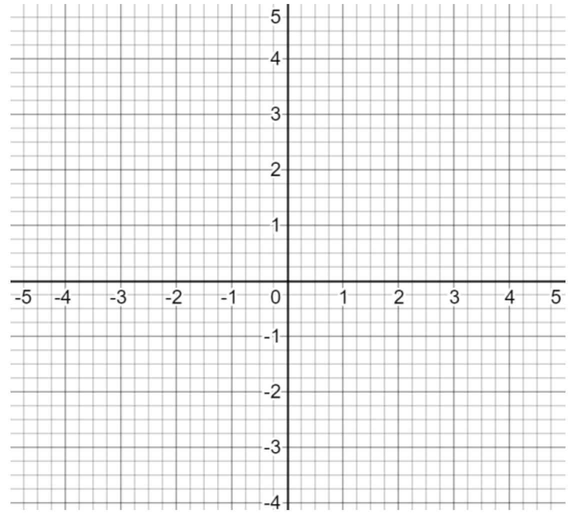
b) $y = f(-x)$



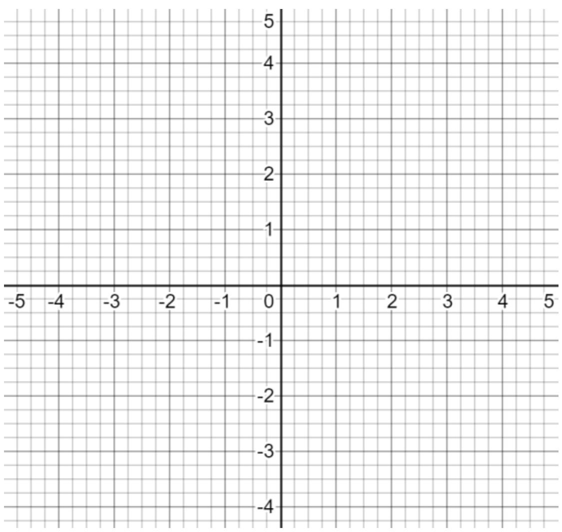
c) $y = -f(-x)$



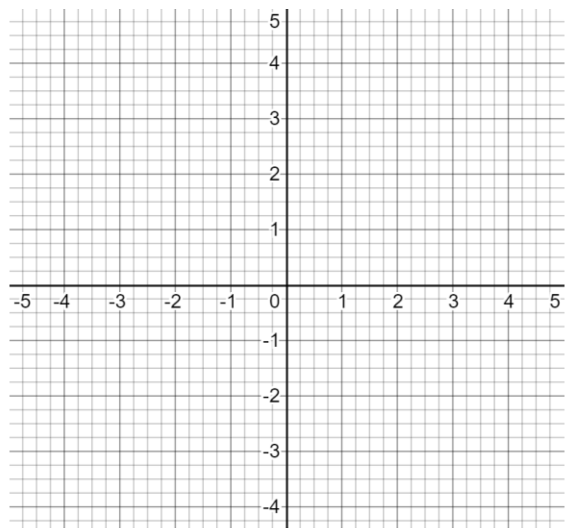
d) $y = f(x+1)$



e) $y = f(x) - 2$



f) $y = f(1-x)$



10. If $(-2, 4)$ is a point on the graph of $y = f(x-1)$, what must be a point on the following graphs?

a) $y = f(x)$

b) $y = -f(x)$

c) $y = f(-x)$

d) $y = f(x) + 2$

e) $y = f(x + 2)$

f) $y = -f(-x)$

11. What is the range of the Absolute Value Function: $f(x) = \left|4 - x^2\right|$

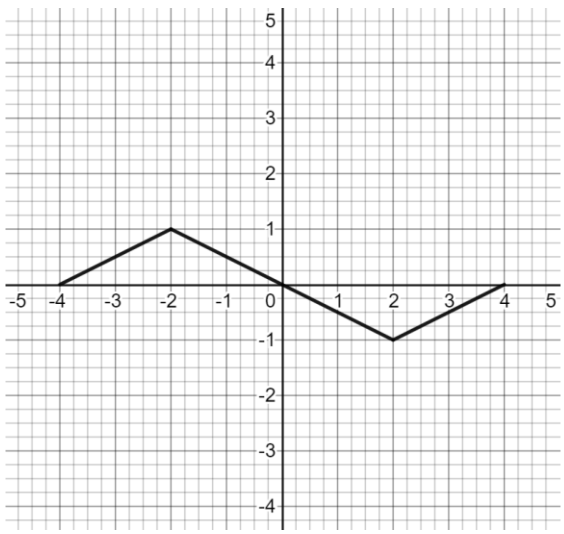
12. If the point $(-1, -2)$ is on the graph $y = f(x)$, what point is on the graph $y = |f(-x)|$?

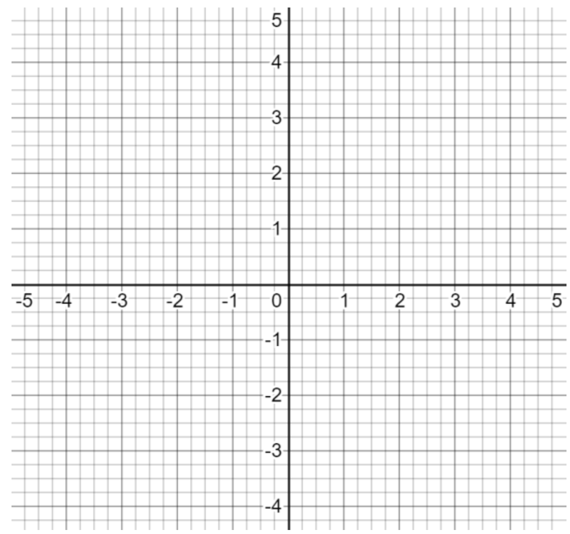
13. If the range of $y = f(x)$ is $-3 \leq y \leq 1$, what is the range of $y = |f(x)|$?

14. If the point $(-3,-6)$ is on the graph $y = f(x)$, what point is on the graph $y = 3|f(x)| + 1$?

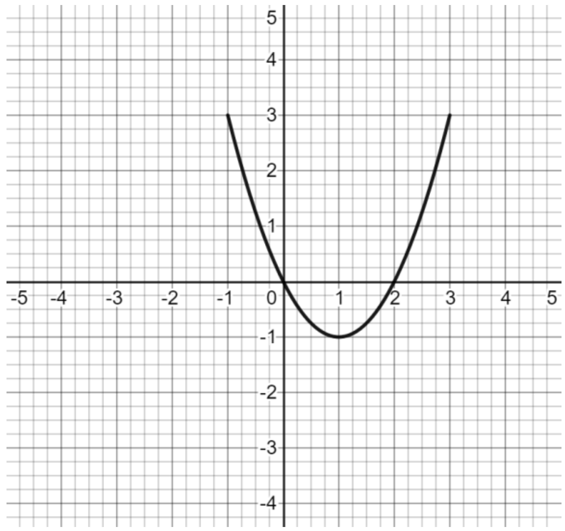
15. Given the graph of $y = f(x)$, graph the reciprocal function $y = \dfrac{1}{f(x)}$

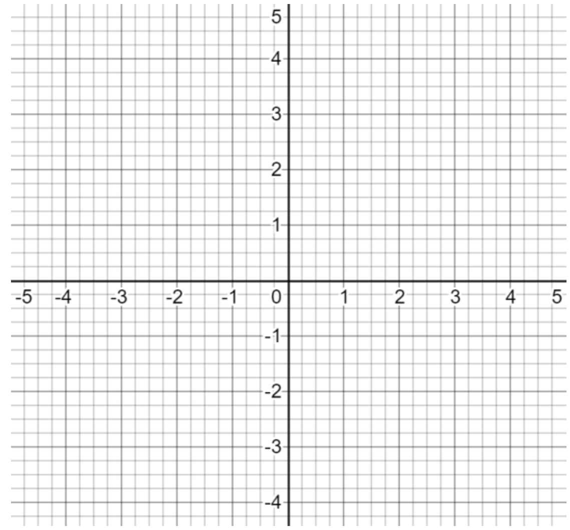
a)





b)

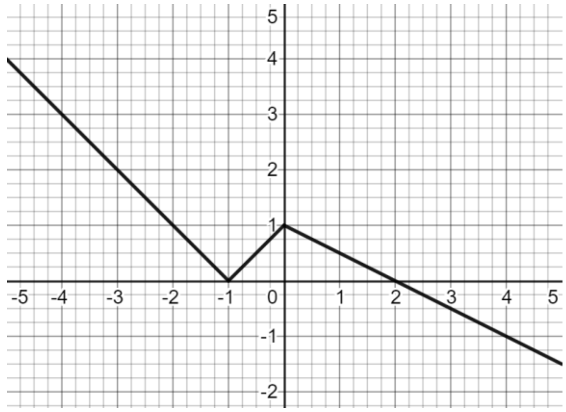




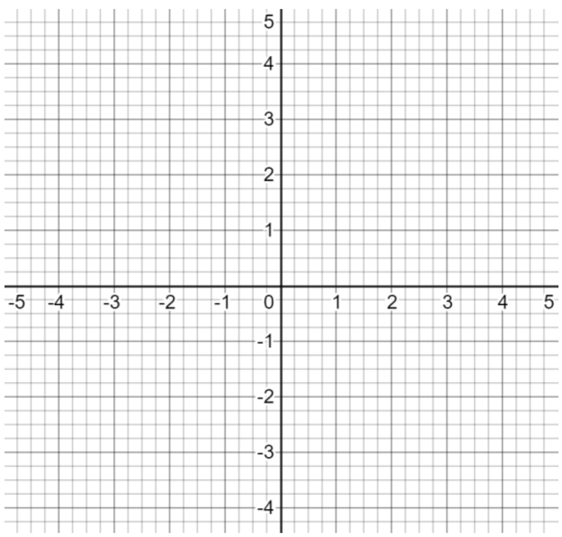
16. If $f(x) \geq 1$, what is the reciprocal function $\frac{1}{f(x)}$ value?

17. If the graph of $y = f(x)$ has the restriction of $0 < f(x) \leq 1$, what are the restrictions of $y = \frac{1}{f(x)}$?

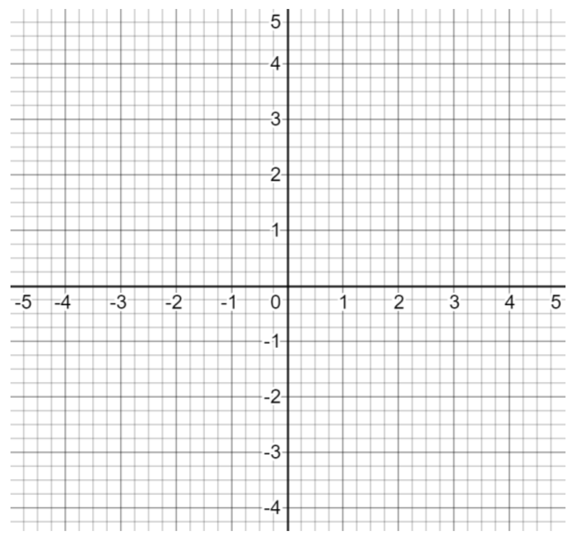
18. Given the graph of $f(x)$ below, sketch the graphs of the following:



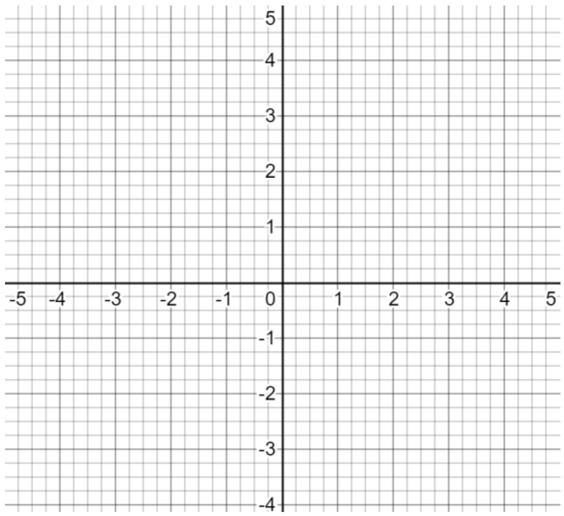
a) $y = 2f(x)$



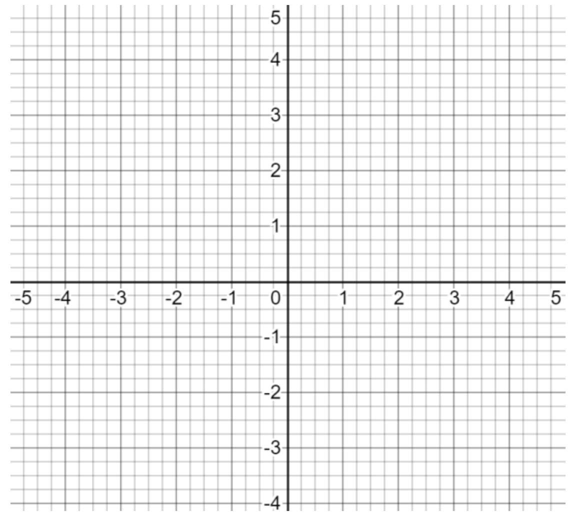
b) $y = f(2x)$



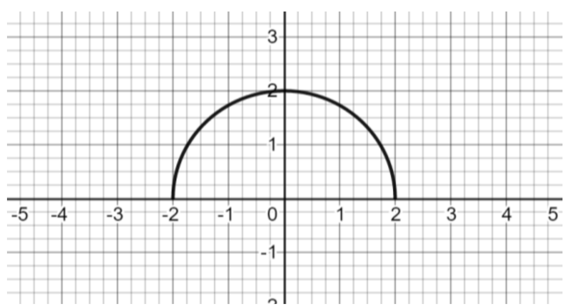
c) $y = -f\left(\frac{x}{2}\right)$



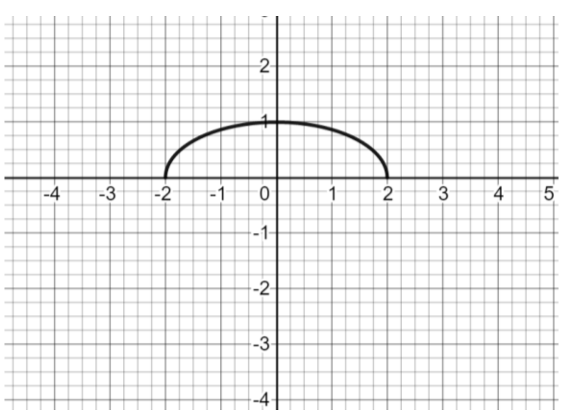
d) $y = -\frac{1}{2}f(-x)$



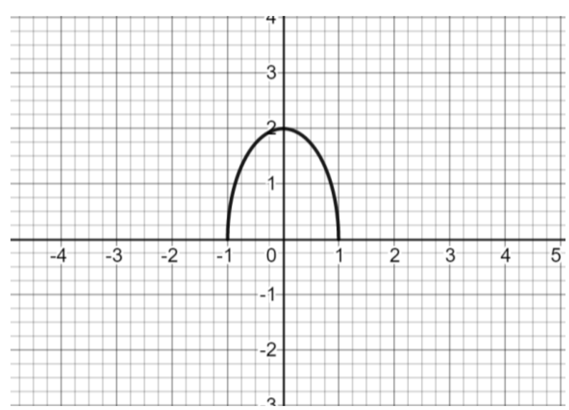
19. Given the graph of $f(x)$ below, what equations represent the following graphs



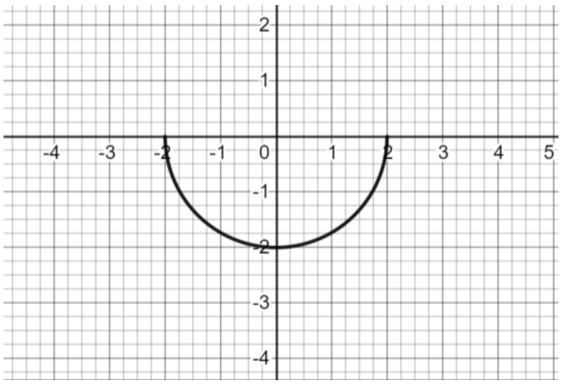
a) $y =$ ____________



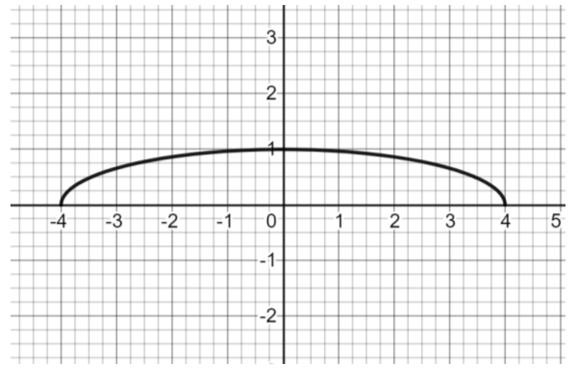
b) $y =$ ____________



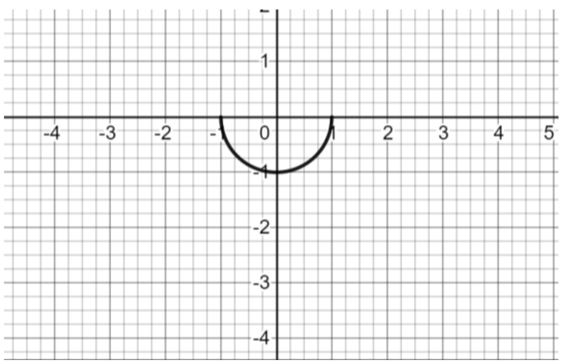
c) $y =$ ____________



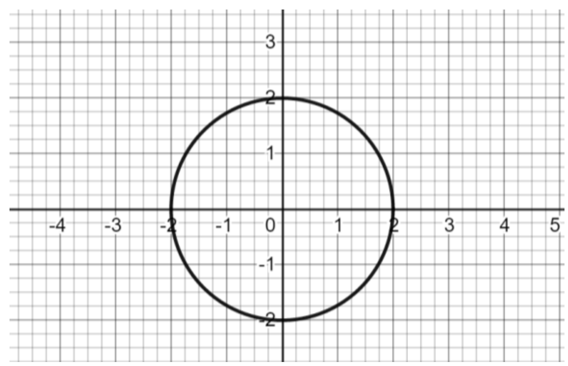
d) $y =$ ____________



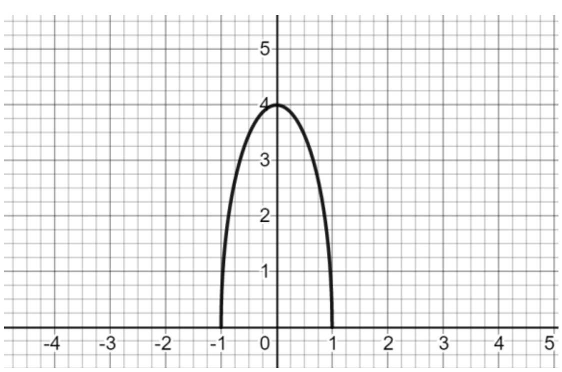
e) $y =$ ____________



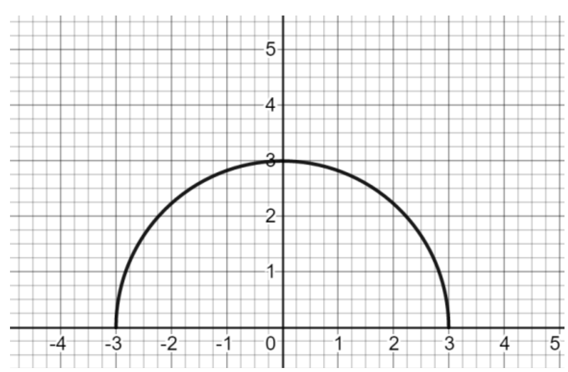
f) $y =$ ____________



g) $y =$ ____________



h) $y =$ ____________



See Website for Detailed Answer Key

Extra Work Space