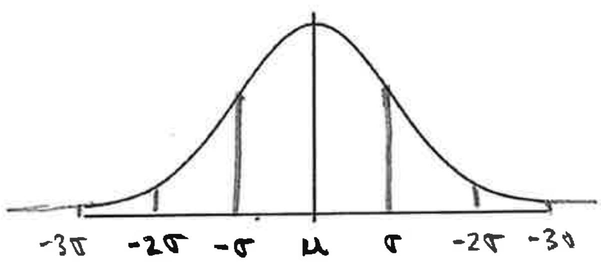
Section 2.3 – Normal Distribution

Normal Distribution

- If you gathered a bunch of data from experiment like Heights, Weights, Tests Scores, etc, It would follow a set pattern. This pattern is called the **Normal Distribution.** It is important because you only need the **Standard Deviation** and the **Mean** to complete the entire distribution

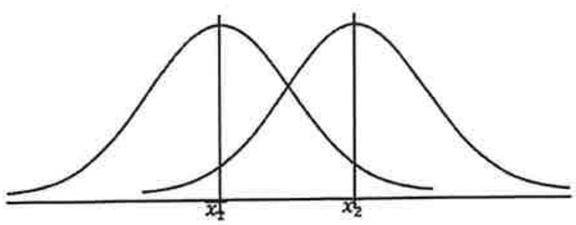
Important Characteristics of the Normal Curve



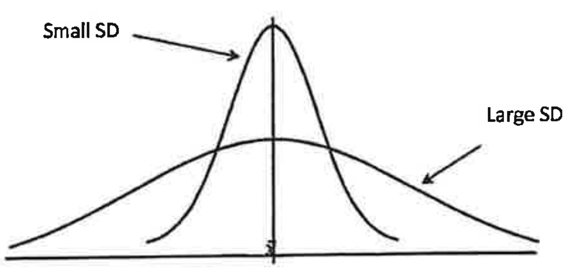
- It is bell shaped and symmetric about the mean
- The enclosed area always equals 1
- The proportion of a population with a certain characteristic or the probability of an event occurring equals the area under the normal curve
- The curve will never touch the x-axis, but extends infinitely in both directions
- The Mean, Median, and Mode are always the same

Percent of population within:

1 standard deviation is 68.2%
2 standard deviations is 95.4%
3 standard deviations is 99.7%



- Two Normal Curves
- They have the SAME Standard Deviation
- But DIFFERENT Means



- Two Normal Curves
- They have the SAME Mean
- But DIFFERENT Standard Deviations (SD)

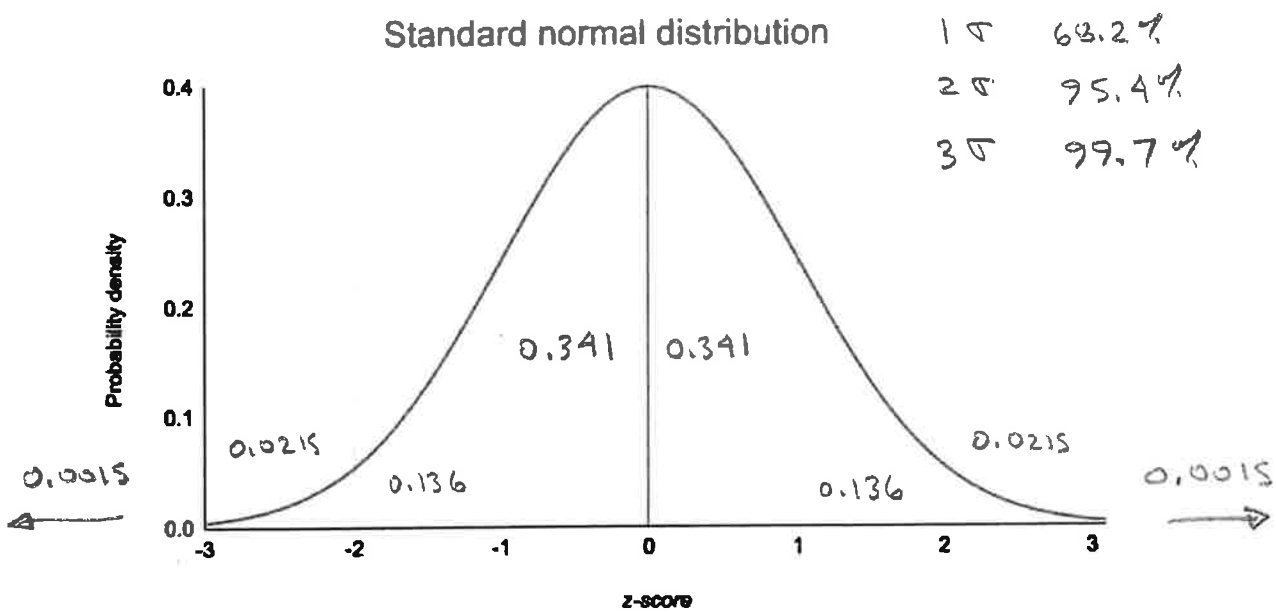
Foundations of Math 11

All of these normal distributions follow the function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

To calculate the proportion of a population with a certain characteristic or the probability of an event occurring, we need to determine the area under the curve. This requires the evaluation of a very difficult integral requiring calculus. Instead of this headache, we use what is called the Standard Normal Distribution Curve and z-scores.

Standard Normal Distribution Curve



The Standard normal distribution curve has a mean, μ = 0 and a standard deviation σ = 1. It uses the formula

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x)^2}$$

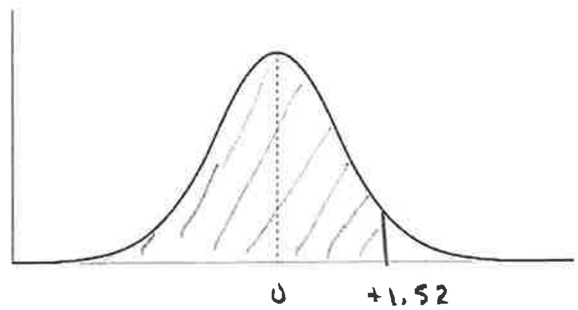
to calculate the area under the curve. The calculations have been done for us and we are given a table to look up the areas under the curve.

Foundations of Math 11

Example: Find the area under the Standard Normal Distribution Curve

a) Left of z = 1.52

$P(z<1.52) = 0.9357$



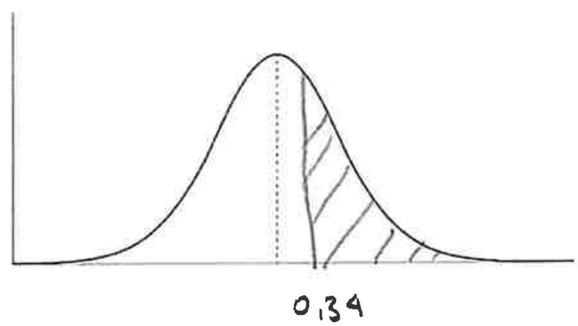
b) Right of z = 0.34

$P(z > 0.34)$

$= 1 - P(z < 0.34)$

$= 1 - 0.6331$

$= 0.3669$



c) Between z = -0.65 and z = 0.78

$P(-0.65 < z < 0.78)$

$= P(z < 0.78) - P(z < -0.65)$

$= 0.7823 - 0.2578$

$= 0.5245$

Practice Problems #1- 5

Z-Scores and Standard Normal Curve

Since there are many different possible Normal Distribution Curves, each with different Means and Standard Deviations, we convert values of the population to a $Z-$score with a transformation.

The Z-Score will fit the Standard Normal Curve with $\mu = 0$ and $\sigma = 1$

We can then look up the area with our $Z-$score table.

Definition

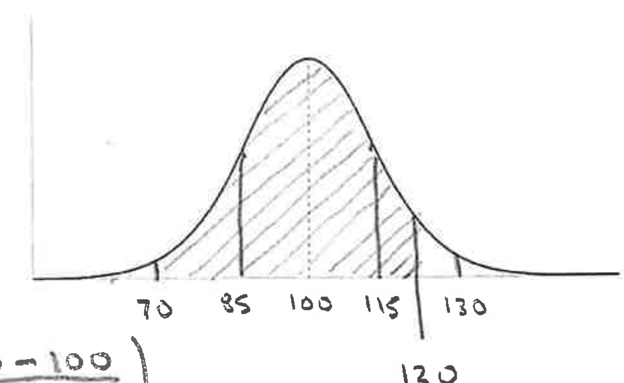
$$Z = \frac{difference\ between\ x\ and\ \mu}{standard\ deviation} = \frac{x-\mu}{\sigma}$$

x: a particular value in the population
μ: the population mean
σ: the population standard deviation
Z: number of Standard Deviations that x is away from μ

Note: There are both negative and positive Z-Scores (See the Table Provided with Z-Scores)

Example: If IQ scores are normally distributed with a Mean 100 and Standard Deviation of 15, determine: the probability that a randomly selected person has an IQ less than 120.

$\mu = 100$
$\sigma = 15$



$$P(x < 120) = P\left(Z < \frac{120-100}{15}\right)$$
$$= P(Z < 1.333)$$
$$= 0.9082$$

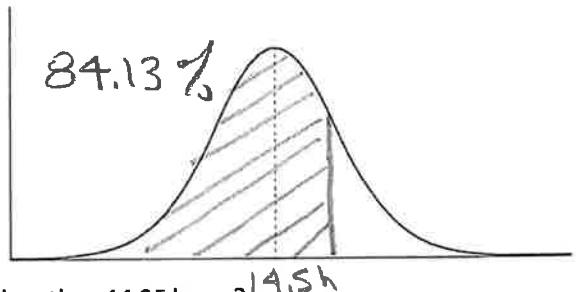
This means that
1. 90.82% of a popn have IQ less than 120
2. Prob that randomly selected person has IQ less than 120 is 90.82%.

Example: The average flight time from Sydney to Los Angeles is 14.5 hours with a standard deviation of 0.5 hours.

$\mu = 14.5h$
$\sigma = 0.5h$

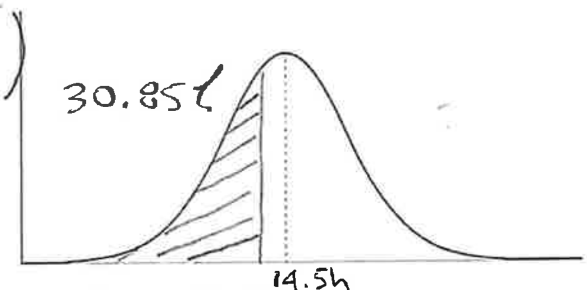
a) What is the probability of a flight less than 15 hours?

$$P(x<15) = P\left(z < \frac{15-14.5}{0.5}\right)$$
$$= P(z<1)$$
$$= 0.8413$$



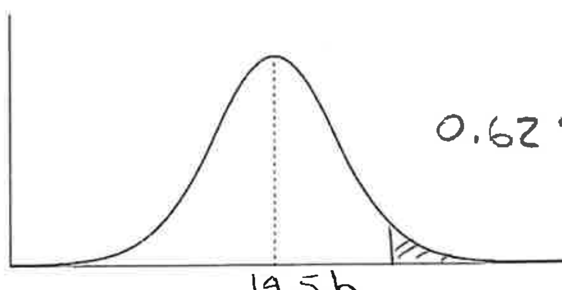
b) What is the probability that the flight is less than 14.25 hours?

$$P(x<14.25) = P\left(z < \frac{14.25-14.5}{0.5}\right)$$
$$= P(z < -0.5)$$
$$= 0.3085$$



c) What is the probability that the flight is longer than 15.75 hours?

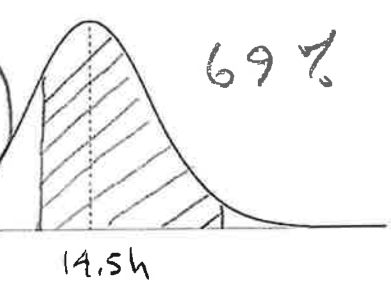
$$P(x>15.75) = 1 - P(x<15.75)$$
$$= 1 - P\left(z < \frac{15.75-14.5}{0.5}\right)$$
$$= 1 - P(z<2.5)$$
$$= 1 - 0.9938 = 0.0062$$



0.62 %

d) What is the probability that the flight is between 14.2 and 15.4 hours?

$$P(14.2 < x < 15.4) = P(x<15.4) - P(x<14.2)$$
$$= P\left(z < \frac{15.4-14.5}{0.5}\right) - P\left(z < \frac{14.2-14.5}{0.5}\right)$$
$$= P(z<1.8) - P(z<-0.6)$$
$$= 0.9641 - 0.2743$$
$$= 0.6898$$



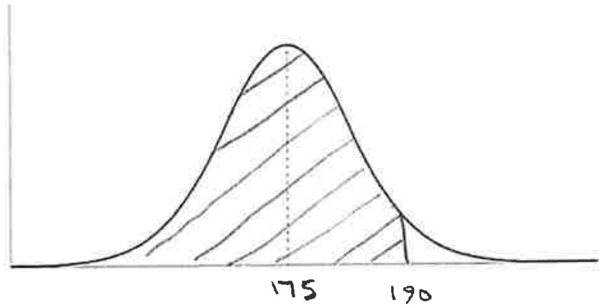
69 %

Practice Problems #6 - 8

Example: The average height of humans in Victoria is 175 cm with a standard deviation of 10 cm.

a) What percent of the population will have a height less than 190 cm?

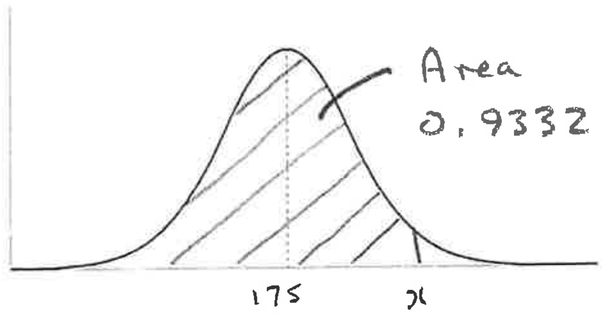
$$P(X<190) = P\left(Z < \frac{190-175}{10}\right)$$
$$= P(Z < 1.50)$$
$$= 0.9332$$
$$= 93.32\% \text{ of popn}$$



b) What height is 93.32 percent of the population?

$$P(X<x) = P\left(Z < \frac{x-175}{10}\right)$$
$$P\left(Z < \frac{x-175}{10}\right) = 0.9332$$
$$\frac{x-175}{10} = Z_{0.9332}$$

(look up Z score with this area)



Using the Z-Score Table Provided (Always indicates the area or probability to the LEFT of the Z-Score)

We see that:

- The closest score to 0.9332 is 0.9332
- Read the table backwards to find the corresponding value
- The Z-Score for 0.9332 is 1.50

- So,

$$\frac{x-175}{10} = Z_{0.9332} \qquad \frac{x-\mu}{\sigma} = Z$$

$$\frac{x-175}{10} = 1.50$$

$$x - 175 = 15$$

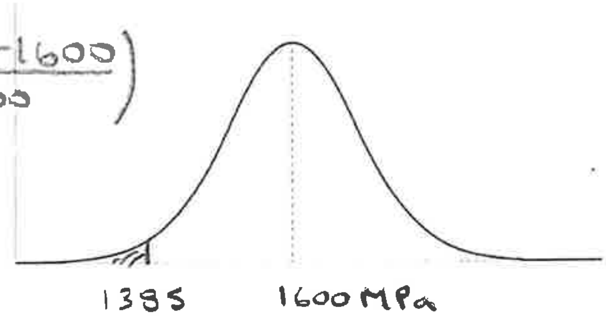
$$x = 190$$

93.32% of the population has height less than 190 cm

Example: Tensile strength is the amount of force per unit area that a bolt can withstand before breaking. The mean tensile strength of a bolt is 1600 MPa with a standard deviation of 100 MPa.

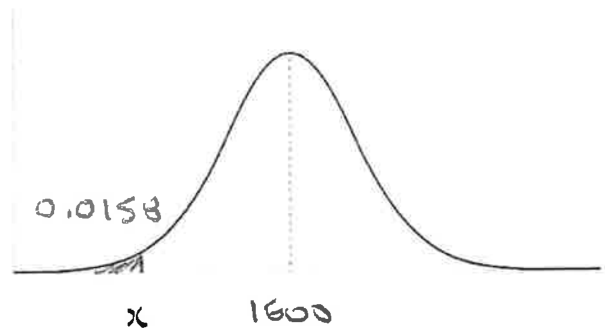
a) What percent of balls would have a strength of less than 1385 MPa and would have to be rejected?

$$P(x < 1385) = P\left(z < \frac{1385-1600}{100}\right)$$
$$= P(z < -2.15)$$
$$= 0.0158$$
$$= 1.58\%$$



b) If 1.58% of bolts are rejected, what is the minimum tensile strength?

$$\frac{x-\mu}{\sigma} = z$$



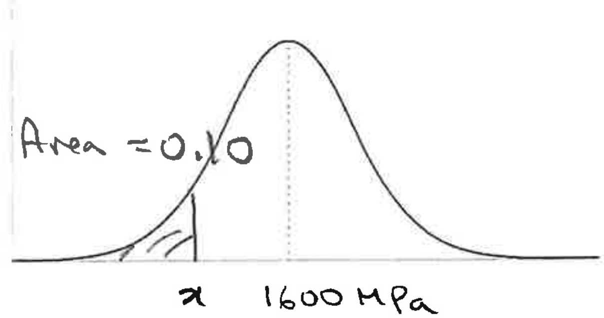
$$\frac{x-1600}{100} = z_{0.0158}$$

(look up z-score with area 0.0158)

$$\frac{x-1600}{100} = -2.15$$
$$x - 1600 = -2.15(100)$$
$$x - 1600 = -215$$
$$x = -215 + 1600$$
$$x = 1385 \text{ MPa}$$

c) If 10% of bolts are allowed to be rejected, what is the minimum tensile strength allowed?

$$\frac{x - \mu}{\sigma} = z$$



$$\frac{x - 1600}{100} = z_{0.10}$$

$$\frac{x - 1600}{100} = -1.28$$

$$x - 1600 = -128$$

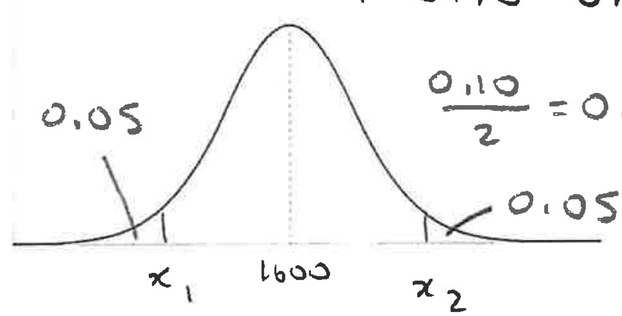
$$x = 1600 - 128$$

$$x = 1472 \text{ MPa}$$

d) What range of strengths, symmetric about the mean would you expect 90% of the bolts to have?

$$1 - 0.90 = 0.10$$

$$\frac{0.10}{2} = 0.05$$



$$\frac{x_1 - \mu}{\sigma} = z_{0.05}$$

$$\frac{x_1 - 1600}{100} = -1.64$$

$$x_1 = 1436 \text{ MPa}$$

$$\frac{x_2 - \mu}{\sigma} = z_{0.95}$$

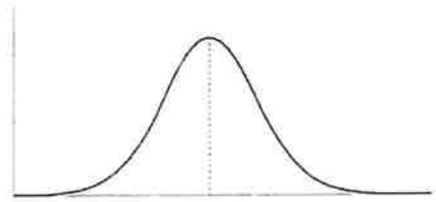
$$\frac{x_2 - 1600}{100} = 1.64$$

$$x_2 = 1764 \text{ MPa}$$

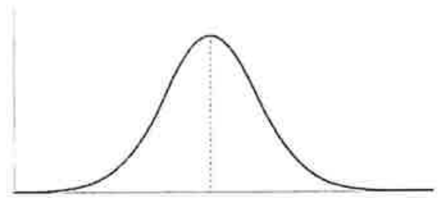
Section 2.3 – Practice Problems

Find the Area under the Standard Normal Curve

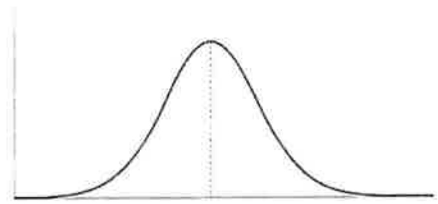
1. Between $z = -0.62$ and $z = 0.75$



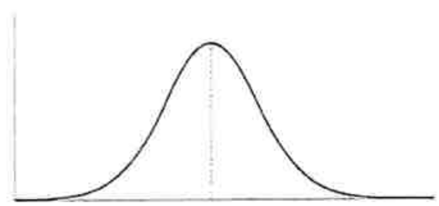
2. Between $z = -2.35$ and $z = 1.42$



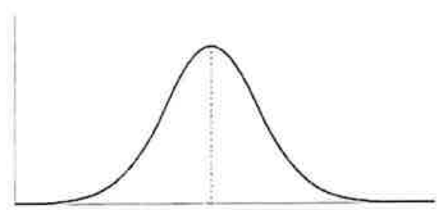
3. Between $z = -1.42$ and $z = -2.38$



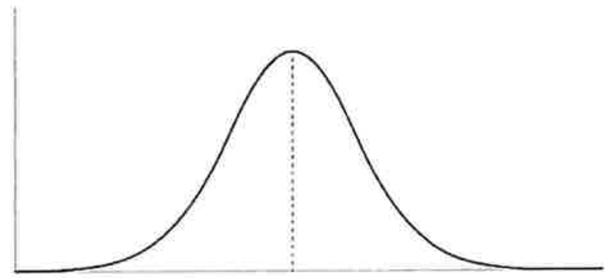
4. To the right of $z = 1.46$



5. To the right of $z = -2.37$

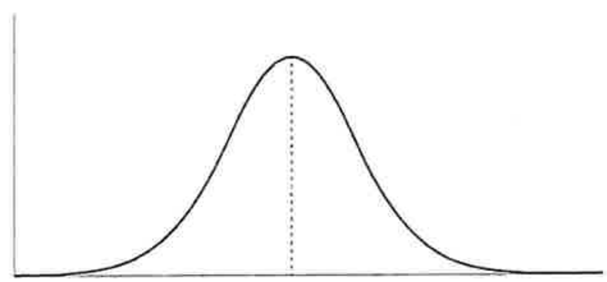


6. The attendance for a week at the local theatre is normally distributed with a mean of 4000 and a standard deviation of 500. What percent of attendance figures fall between 3600 and 4600?



7. The average height of humans in Victoria is 175 cm with a standard deviation of 10 cm.

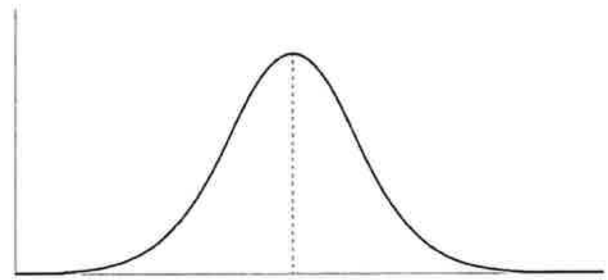
a) How many people would have a height between 160 cm and 180 cm?



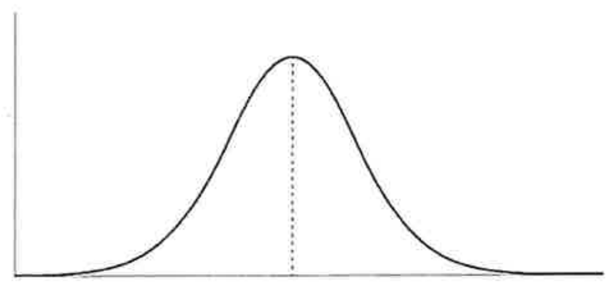
b) If you were to random choose a person, what is the probability that they have a height between 160 cm and 180 cm?

c) If the population of Victoria is 86 000, how many would you expect to have a height between 160 cm and 180 cm?

8. A provincial math exam has a mean of 68 and a standard deviation of 13.2. If 30000 students take the exam, and a score of 49 or less fails, how many students fail the exam?

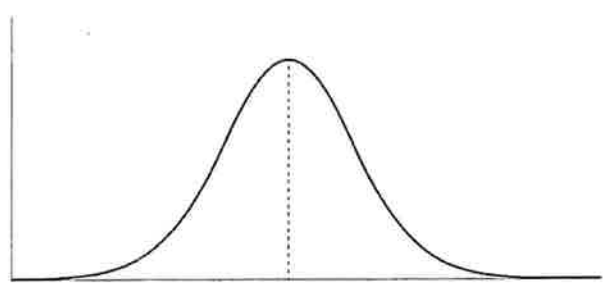


9. A manufacture of iPhone indicated a mean of 26 months before there is need of repairs with a standard deviation of 6 months. What length of time for the warranty should the manufacture set such that less that 10% of all iPhone will need repairs during this warranty period?

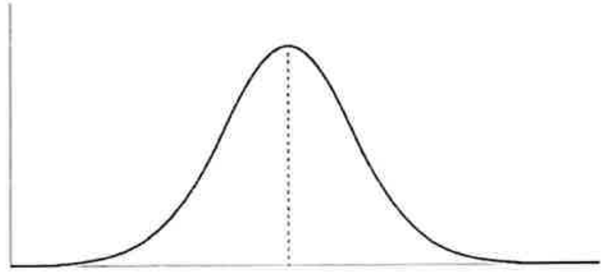


10. Nylon strands are manufactured to a mean tensile strength of 1.5 N, with a standard deviation of 0.04 N. If the tensile strength is normally distributed,

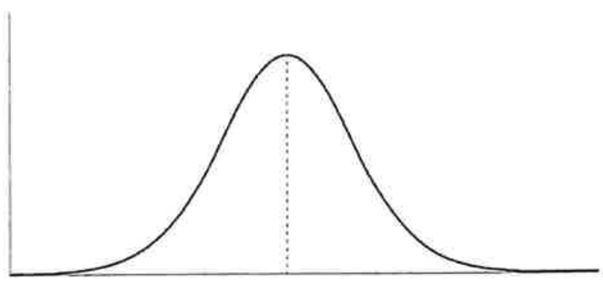
a) what percent of the strands would have a strength less than 1.4 N?



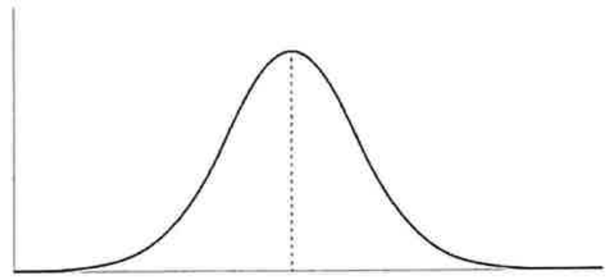
b) what range of strengths, symmetrical about the mean, would you expect 99.0% of the strands to have?



11. The accuracy of an automatic pitching machine for batting practice is based on the off-line distance that a pitch is from a target line that is 30 m away. The off-line distance is normally distributed with a mean of 0.3 m and a standard deviation of 0.05 m. What percent of the pitches fall within 0.2 m of the target line?

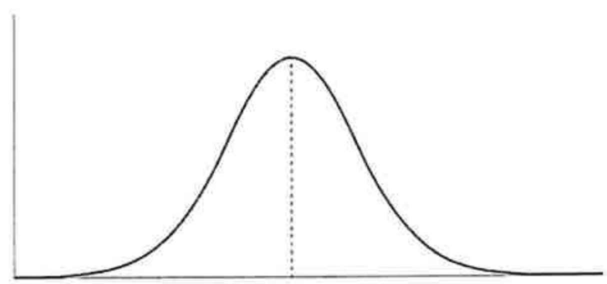


12. Major manufacturing companies operate on the principle of preventive maintenance to avoid a complete shutdown of the assembly line if a component fails. The lifetime of one component is normally distributed with a mean of 321 h an d a standard deviation of 23 h. How frequently should the component be replaced so that the probability of its failing during operation is less than 0.001?

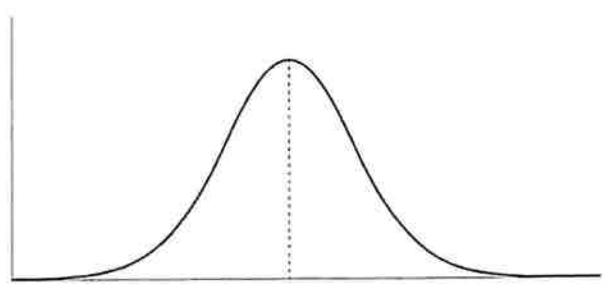


13. The Trans-Canada highway stretches from St. Hon's to Victoria. On one section of the highway it has been found that motorists drive at speeds that are normally distributed with a mean of 110 km/h and a standard deviation of 16 km/h.

a) What percent of motorists are driving less than or at the posted speed limit of 100 km/h on this section?

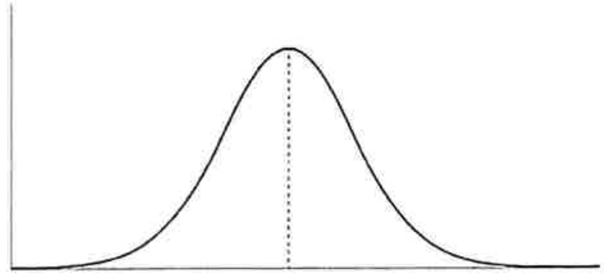


b) Below what speed do 90% of the motorists drive?

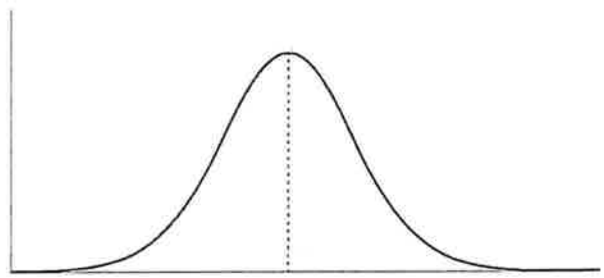


14. Students' marks on a test were normally distributed with a mean of 70 and a standard deviation of 8

a) What percent of the students obtained a mark above 80?



b) What percent of the students obtained a C grade (60 to 70)?



c) Determine the mark under which 75% of the students' marks occur. This is referred to the 75th percentile