Section 2.3 – Composite Functions

- There is another way of combining functions, the **COMPOSITION OF A FUNCTION**
- We use the process of substitution, we substitute an entire function into another given one

$$(f \circ g)(x) \quad \textbf{means} \quad f \text{ of } g \text{ of } x, \ \textbf{ or } \ f \textbf{ composed with } g$$

$$(f \circ g)(x) \quad \rightarrow \quad f(g(x))$$

Given the two function $f(x) = x^2$ and $g(x) = 2x - 1$

Composition in the Abstract	*Composition using a Specific Value*
$(f \circ g)(x)$	$(f \circ g)(x)$ **when** $x = 3$
$(f(g(x))$	$f(g(x)) \rightarrow f(g(3))$
$x^2 \rightarrow (2x-1)^2$	$g(3) = 2(3) - 1 = 5$
This is $f(x)$ (points to x^2)	**so** … $f(g(3)) = f(5)$
This is $f(x)$ with the $g(x)$ function subbed in for the x in the $f(x)$ function (points to $(2x-1)^2$)	5^2
	25

We use this formal definition:

Composite of Functions $(f \circ g)$

The composite function $(f \circ g)$ of the two functions f *and* g is defined by $(f \circ g)(x) = f(g(x))$.

For all x in the Domain of g such that $g(x)$ is in the Domain of f.

Example 1: If $f(x) = 1 - x^2$, $g(x) = 2x + 3$, find

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$

Solution 1:

a) $(f \circ g)(x) = f(g(x))$

$= f(2x + 3)$ — Substitute the $g(x)$ for its function

$= 1 - (2x + 3)^2$ — Substitute the $g(x)$ function into the x in the $f(x)$ function

$= 1 - (2x + 3)(2x + 3)$

$= 1 - (4x^2 + 12x + 9)$ — Do not forget to FOIL

$= \mathbf{-4x^2 - 12x - 8}$ — Simplify

b) $(g \circ f)(x) = g(f(x))$ — Substitute the $f(x)$ for its function

$= g(1 - x^2)$ — Substitute the $f(x)$ function into the x in the $g(x)$ function

$= 2(1 - x^2) + 3$

$= 2 - 2x^2 + 3$ — Do not forget to WATERBOMB

$= \mathbf{-2x^2 + 5}$ — Simplify

Example 2: If $f(x) = x^2$ *and* $g(x) = 2x - 1$, find $(f \circ g)(-2)$

Solution 2:

Method 1: Abstract 1st	Method 2: Piecewise
$(f \circ g)(x) = f(g(x))$ $= f(2x - 1)$ $= (2x - 1)^2$ $= 4x^2 - 4x + 1$ So… $(f \circ g)(-2)$ $= 4(-2)^2 - 4(-2) + 1$ $= 16 + 8 + 1$ $= 25$	$g(x) = 2x - 1$ $g(-2) = 2(-2) - 1$ $g(-2) = -5$ So… $(f \circ g)(-2) = f(g(-2))$ $= f(-5)$ $= (-5)^2$ $= 25$

Example 3: If $f(x) = x^2 - 1 \quad and \quad g(x) = \sqrt{x}$, find $(f \circ g)(x)$, the Domain of $(f \circ g)(x)$, and sketch the graph

Solution 3: Start by identifying the Domains of the individual functions

Domain of $f(x)$: $\boldsymbol{x = All\ Real\ Numbers}$

Domain of $g(x)$: $\boldsymbol{x \geq 0}$

$$(f \circ g)(x) = f(g(x))$$

$$= f(\sqrt{x})$$

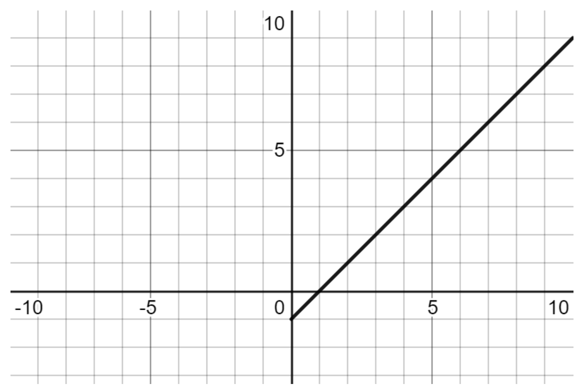
$$= \sqrt{x}^2 - 1$$

$$= \boldsymbol{x - 1}$$

The Domain of $(f \circ g)(x)$ appears to be:

$x = All\ Real\ Numbers,$

but it is restricted by the Domain of $g(x)$.



Example 4: If $f(x) = \dfrac{x}{x-1}$ and $g(x) = \dfrac{1}{x+1}$, find

a) $(f \circ g)(x)\ and\ its\ Domain$ b) $(g \circ f)(x)\ and\ its\ Domain$

Solution 4:

a) $(f \circ g)(x) = f(g(x))$

$$= f\left(\frac{1}{x+1}\right)$$

$$= \frac{\frac{1}{x+1}}{\frac{1}{x+1} - 1} = \frac{\frac{1}{x+1}}{\frac{1-x-1}{x+1}}$$

$$= \frac{1}{x+1} \cdot \frac{x+1}{1-x-1}$$

$$= \frac{1}{x+1} \cdot \frac{x+1}{-x} = \frac{1}{-x} = -\frac{1}{x}$$

- Common denominator
- Fractions of Fractions can get tricky
- Helps to Flip and Multiply
- Keeps things clean

The Domain of $g(x) = \dfrac{1}{x+1}$ is, $x \neq -1$

The Domain of $(f \circ g)(x)$ is, $x \neq 0$

So, the **Domain of $\boldsymbol{(f \circ g)(x)}$ is $\boldsymbol{x \neq 0, -1}$**

b) $(g \circ f)(x) = g(f(x))$

$$= g(\frac{x}{x-1})$$

$$= \frac{1}{\frac{x}{x-1}+1} = \frac{1}{\frac{x+x-1}{x-1}}$$

$$= \frac{1}{1} \cdot \frac{x-1}{2x-1}$$

$$= \frac{x-1}{2x-1}$$

- Common Denominator
- Fractions of Fractions can get tricky
- Helps to Flip and Multiply
- Keeps things clean

The Domain of $f(x) = \dfrac{x}{x-1}$ is, $x \neq 1$

The Domain of $(g \circ f)(x)$ is, $x \neq \dfrac{1}{2}$

So, the **Domain of** $\boldsymbol{(g \circ f)(x)}$ **is** $\boldsymbol{x \neq 1, \dfrac{1}{2}}$

Example 5: If $f = \{(1, d), (3, e)\}$ and $g = \{(a, 1), (b, 3), (c, 5)\}$, find $\boldsymbol{(f \circ g)(x)}$

Solution 5: Need to run the inputs from function g and match their outputs to the inputs from function f to get the output from f as a solution.

$$\boldsymbol{(f \circ g)(a)} = f\big(g(a)\big) = f(1) = \boldsymbol{d}$$

$$\boldsymbol{(f \circ g)(b)} = f\big(g(b)\big) = f(3) = \boldsymbol{e}$$

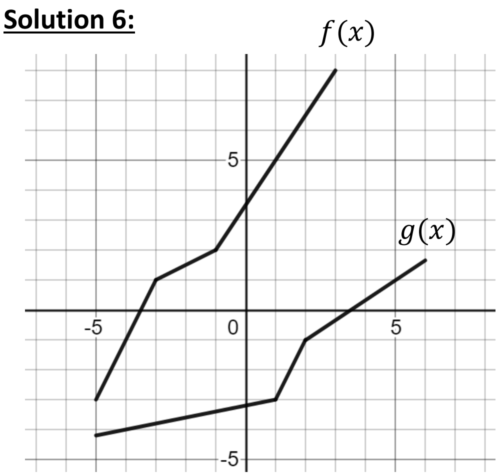
$$\boldsymbol{(f \circ g)(c)} = f\big(g(c)\big) = \boldsymbol{f(5)\ but\ 5\ is\ not\ in\ the\ Domain\ of\ f(x)},$$

$$\boldsymbol{so\ (f \circ g)(c)\ cannot\ be\ found}$$

Therefore, $\boldsymbol{(f \circ g)(x)} = \{(\boldsymbol{a}, \boldsymbol{d}), (\boldsymbol{b}, \boldsymbol{e})\}$

Example 6: Use the graph to find a) $(f \circ g)(1)$ and b) $(g \circ g)(5)$

Solution 6:



a) $\boldsymbol{(f \circ g)(1)} = f(g(1))$

$$= f(-3)$$

$$= \mathbf{1}$$

b) $\boldsymbol{(g \circ g)(5)} = g(g(5))$

$$= g(1)$$

$$= \mathbf{-3}$$

Example 7: Compute: $\frac{f(x+h)-f(x)}{h}\quad h \neq 0\ for\ f(x)=2x^2+3$

Solution 7: This is a technique, and particular equation that is used as the generic form for calculating limits in Calculus, you will see this in Calculus 12!!

$$f(x)=2x^2+3$$

$$f(x+h)=2(x+h)^2+3$$

So…

$$\frac{f(x+h)-f(x)}{h}=\frac{[2(x+h)^2+3]-[2x^2+3]}{h}$$

$$=\frac{[2(x^2+2xh+h^2)+3]-[2x^2+3]}{h}$$

$$=\frac{2x^2+4xh+2h^2-2x^2-3}{h}$$

$$=\boldsymbol{4x+2h}$$

Decomposing a Composite Function

- When you get comfortable building them up, decomposing them will be quite intuitive
- Consider the **input value (what's on the inside)** and the **output value (what's on the outside)**

Example 8: Given $h(x)=\sqrt{x-2}$, find the two functions $f\ and\ g$ so that $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x})=\boldsymbol{h}(\boldsymbol{x})$

Solution 8:

- The inside: $x-2$ so let: $(x-2)=g(x)$
- The outside: $\sqrt{x}$ and let: $\sqrt{x}=f(x)$

Check:

$$(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))=\boldsymbol{f}(\boldsymbol{x}-\mathbf{2})$$

$$\sqrt{\boldsymbol{x}-\mathbf{2}}=\boldsymbol{h}(\boldsymbol{x})$$

Example 9: Given $h(x) = \sqrt[3]{x+5}$, find the two functions $f\ and\ g$ so that $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x})$

Solution 9:

- The inside: $x + 5$ so let: $(x+5) = g(x)$
- The outside: $\sqrt[3]{x}$ and let: $\sqrt[3]{x} = f(x)$

Check:

$$(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}\big(\boldsymbol{g}(\boldsymbol{x})\big) = \boldsymbol{f}(\boldsymbol{x} + \mathbf{5})$$

$$\sqrt[3]{\boldsymbol{x} + \mathbf{5}} = \boldsymbol{h}(\boldsymbol{x})$$

Example 10: Given $h(x) = \left(\sqrt{x} + 1\right)^3 - 2$, find the two functions $f\ and\ g$ so that $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x})$

Solution 10:

- The inside: $\sqrt{x} + 1$ so let: $\sqrt{x} + 1 = g(x)$
- The outside: $x^3 - 2$ and let: $x^3 - 2 = f(x)$

Check:

$$(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}\big(\boldsymbol{g}(\boldsymbol{x})\big) = \boldsymbol{f}(\sqrt{x} + 1)$$

$$\left(\sqrt{\boldsymbol{x}} + \mathbf{1}\right)^{\mathbf{3}} - \mathbf{2} = \boldsymbol{h}(\boldsymbol{x})$$

Section 2.3 – Practice Problems

1. What is the Domain of the following functions?

a) $f(x) = \dfrac{1}{x-2}$

b) $f(x) = \dfrac{x-3}{x^2-9}$

c) $f(x) = \sqrt{x+2}$

d) $f(x) = \sqrt{3-x}$

e) $f(x) = \dfrac{1}{\sqrt{x}}$

f) $f(x) = \sqrt{x^2-1}$

g) $f(x) = \sqrt{1-x^2}$

h) $f(x) = \sqrt{x(x-2)}$

2. Let $f(x) = 2x^2 - 3x + 1$, $g(x) = x + 1$, $h(x) = 5$, $j(x) = \dfrac{x-1}{x+1}$

   Evaluate the following

a) $(f \circ g)(2)$

b) $(h \circ j)(-3)$

c) $(j \circ h)(2)$

d) $(j \circ g)(0)$

e) $(h \circ j)(-1)$

f) $(f \circ j)(3)$

g) $(h \circ g \circ g)(2)$

h) $(f \circ f \circ f)(-1)$

i) $(j \circ h \circ g)(-3)$

j) $(g \circ j \circ f)(4)$

k) $(f \circ h \circ j)(2)$

l) $(j \circ j \circ g \circ f)(-2)$

3. Use f and g by the following table of values to evaluate the following:

x	-2	0	3	7
$f(x)$	0	1	4	6

x	-1	1	4	6
$g(x)$	3	2	-2	-4

a) $f(0)$

b) $g(1)$

c) $(f \circ g)(-1)$

d) $(f \circ g)(4)$

e) $(g \circ f)(0)$

f) $(g \circ f)(7)$

g) $(f \circ g)(1)$

h) $(g \circ f)(-2)$

4. For each pair of functions, find $(f \circ g)(x)$ and $(g \circ f)(x)$. State the Domain of the result

a) $f(x) = \sqrt{x+2}$ and $g(x) = 2x - 3$

b) $f(x) = \frac{2}{x}$ and $g(x) = \frac{x}{x-1}$

c) $f(x) = x^2 - 2x$ and $g(x) = x + 3$

d) $f(x) = x^2 + x - 3$ and $g(x) = x + 2$

e) $f(x) = \sqrt{x-2}$ and $g(x) = 3x + 2$

f) $f(x) = |x| - 3$ and $g(x) = -2x + 3$

g) $f(x) = \frac{3}{x}$ and $g(x) = \frac{1}{x-4}$

h) $f(x) = |x-2| - 3$ and $g(x) = \frac{1}{x}$

5. Find two functions $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$. Answers may vary.

a) $h(x) = (2x-3)^2$

b) $h(x) = \sqrt[3]{3x^2 - 2}$

c) $h(x) = \frac{1}{3x-4}$

d) $h(x) = \frac{2}{x^2+4}$

e) $h(x) = \sqrt{x^2+1} + 3$

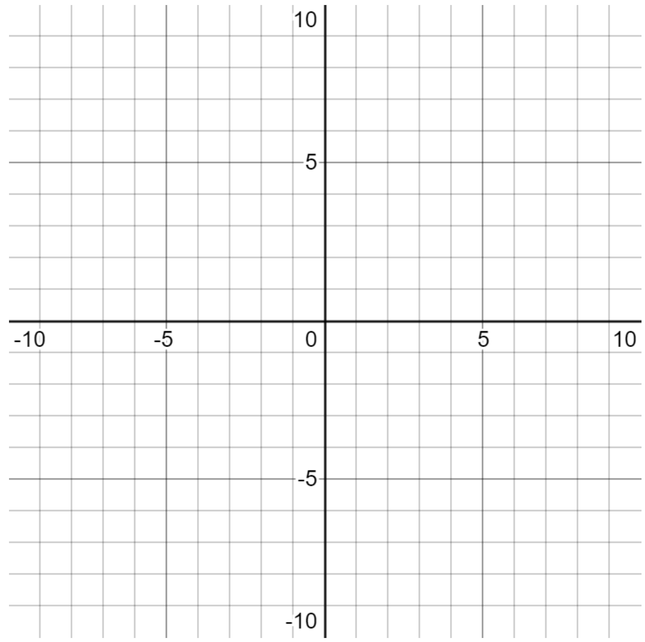
f) $h(x) = \sqrt[3]{3x+4} - 1$

g) $h(x) = 3(2x-3)^4 - (2x-3)^7$

h) $h(x) = 3(2x+4)^3 + 2(2x+4)^6$

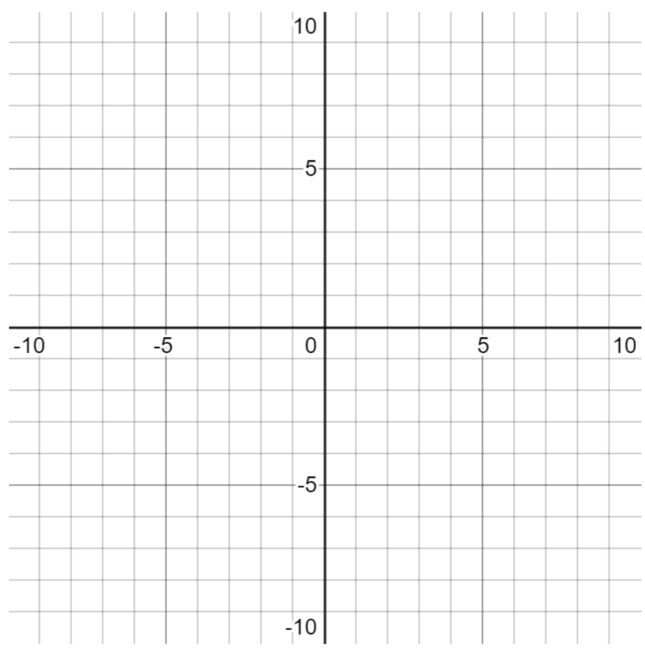
6. Sketch the graph of the $(f \circ g)(x)$ for the following. State the Domain.

a) $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$



$Domain$:

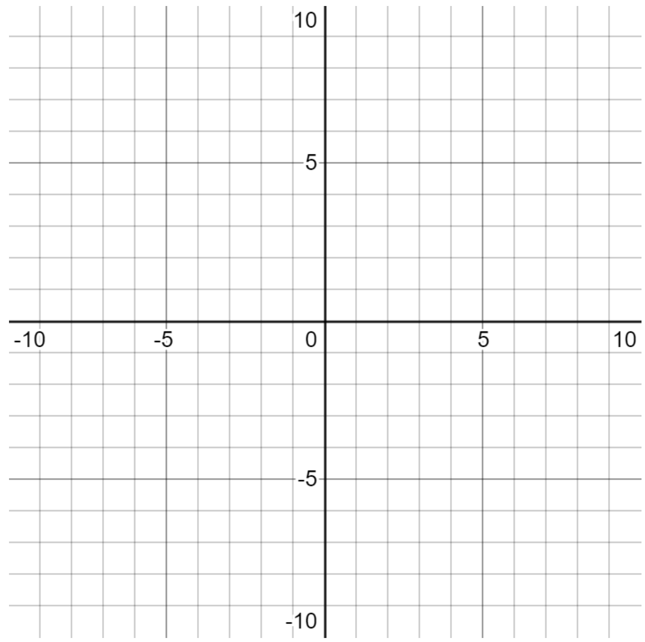
b) $f(x) = x^2 + 2$ and $g(x) = \sqrt{x - 2}$



$Domain$:

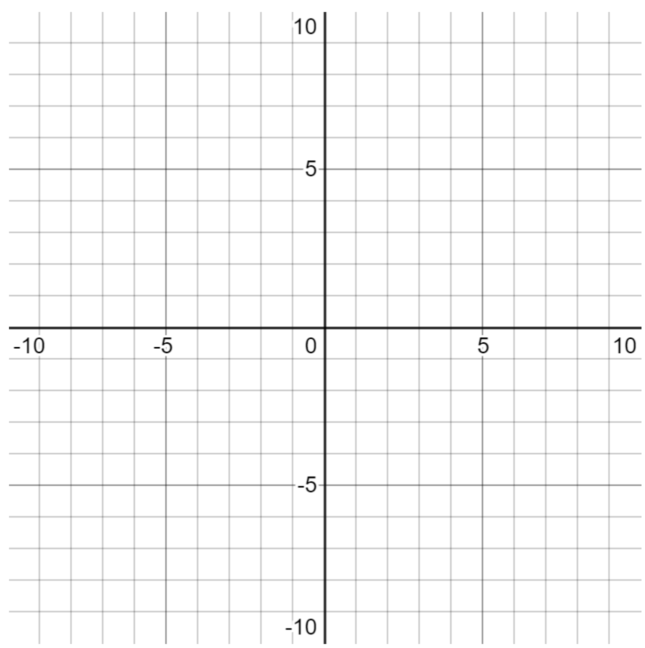
Sketch the graph of the $(f \circ g)(x)$ for the following. State the Domain.

c) $f(x) = 1 - x^2$ and $g(x) = \sqrt{1 - x}$



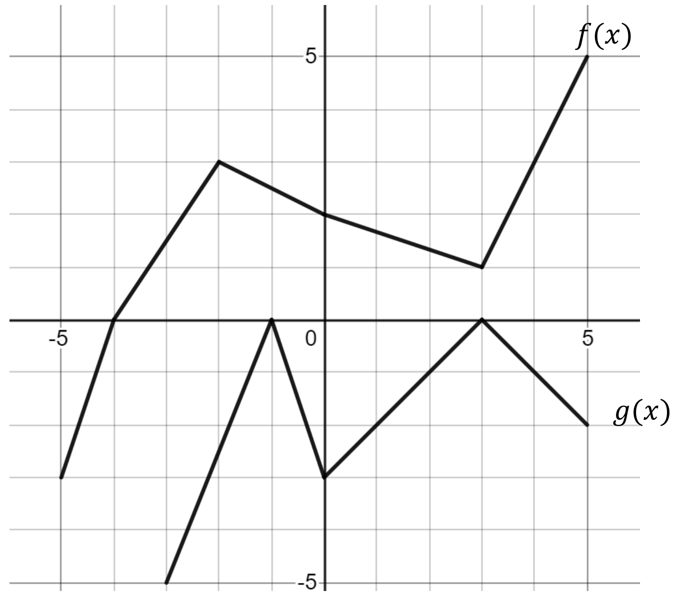
Domain:

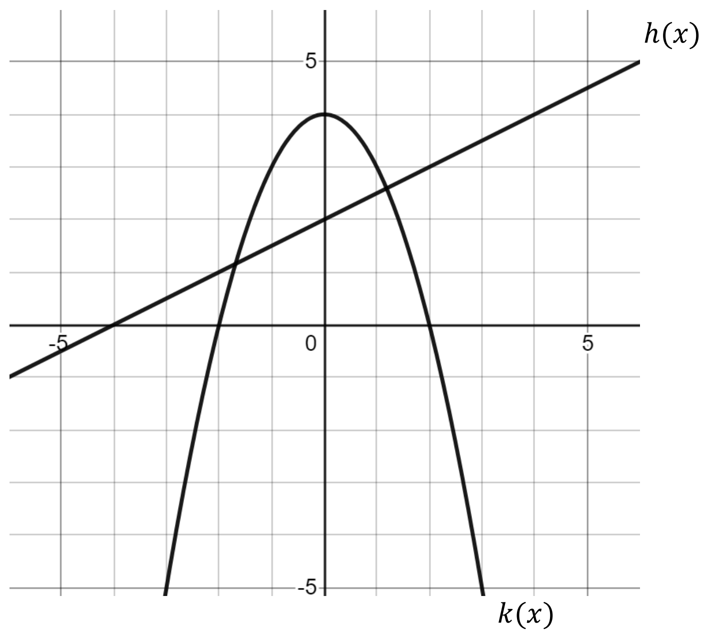
d) $f(x) = x^2 + 1$ and $g(x) = \sqrt{4 - x^2}$



Domain:

7. The first of the two graphs shows two functions $f\ and\ g$. The second shows two functions $h\ and\ k$. Use the graphs to compute the following:





a) $(g \circ f)(-4) =$

b) $(f \circ g)(3) =$

c) $(f \circ f)(-2) =$

d) $(g \circ g)(3) =$

e) $(g \circ f)(-5) =$

f) $(g \circ f)(-3) =$

g) $(h \circ k)(0) =$

h) $(h \circ k)(-1) =$

i) $(h \circ k)(2) =$

j) $(h \circ k)(-3) =$

k) $(k \circ h)(0) =$

l) $(k \circ h)(2) =$

m) $(k \circ h)(-4) =$

n) $(k \circ h)(-2) =$

8. If $f = \{(3,4),(4,5),(5,6),(6,7)\}$ and $g = \{(5,3),(6,4),(7,-2),\ (8,0)\}$, determine:

a) $(f \circ g)(x)$

b) $(g \circ f)(x)$

9. If $f(x) = 3x - 2\ \ and\ \ g(x) = 3x + b$, find b such that $(f \circ g)(x) = (g \circ f)(x)$ for all real numbers x.

10. Find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ $h \neq 0$ for the given function f

a) $f(x) = 2x + 3$

b) $f(x) = x^2 + x$

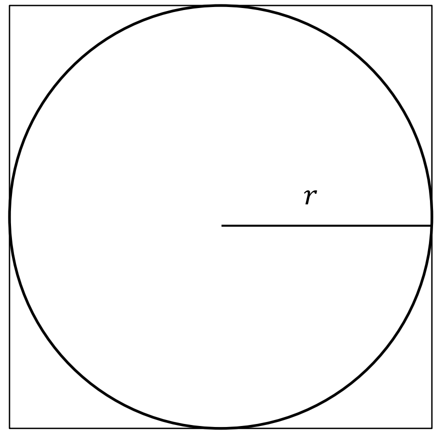
c) $f(x) = -3x^2 + 2x$

d) $f(x) = \dfrac{1}{x}$

e) $f(x) = \frac{4}{2x - 1}$

f) $f(x) = \frac{1}{\sqrt{x}}$

11. A circle inscribed in a square.

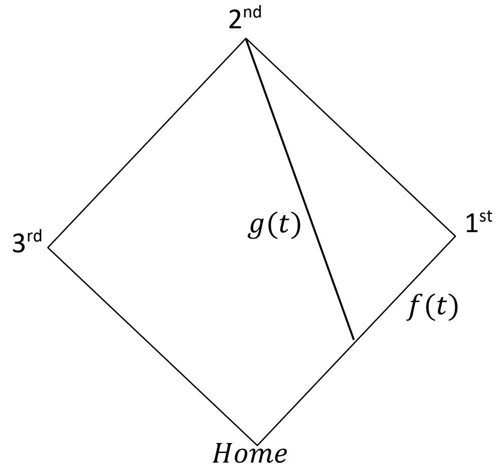


a) Write the radius of the circle as a function of the length x of the sides of the square.

b) Write the area A of the circle as a function of the radius.

c) Find $(A \circ r)(x)$.

12. A baseball diamond is a square $90ft$ on each side. A batter is running to first base at a rate of $27\ ft/sec$



a) Find the function $f(t)$ for the distance x of the batter from first base in terms of time t

b) Find a function $g(f)$ for the distance the batter is from second base in terms of the distance f

c) Find $(g \circ f)(t)$ and explain the meaning of the function.

See Website for Detailed Answer Key

Extra Work Space