Section 10.4 – Practice Problems

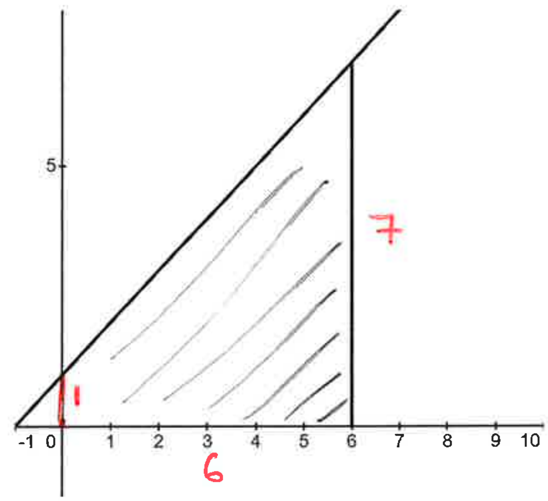
1. R is the region under $y = x + 1$ from 0 to 6

a) Calculate the area of R using the formula for the area of a trapezoid.

$A = \frac{1}{2}(6)(7+1)$

$= 3(8)$

$= \boxed{24}$



b) Approximate the area of R by dividing it into six subintervals of equal width and summing the areas of rectangles.

$f(1) = 2 \quad f(4) = 5$

$f(2) = 3 \quad f(5) = 6$

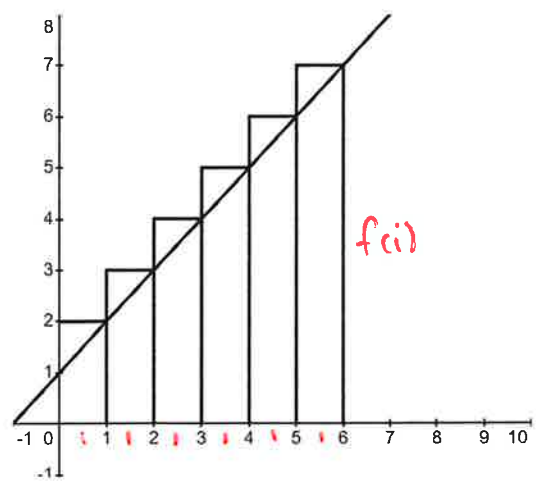
$f(3) = 4 \quad f(6) = 7$

$A = \sum_{i=1}^{6} 1 f(i)$

$= 1(2+3+4+5+6+7)$

$= 1(27)$

$= \boxed{27}$



2. R is the region under $y = x^2 + 1$ from 1 to 3

a) Calculate the area of R using the differential equation $A'(x) = f(x)$.

$f(x) = A'(x)$

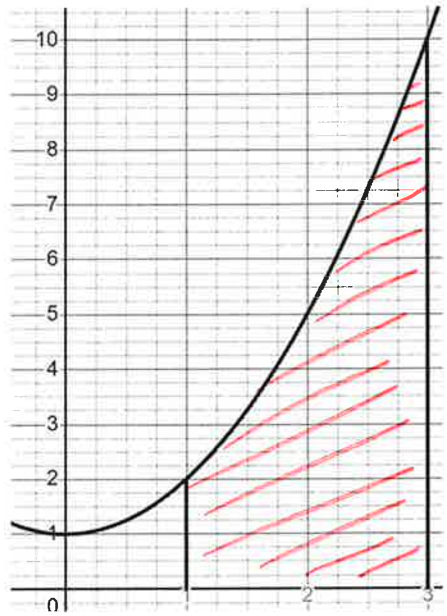
$A(x) = \frac{1}{3}x^3 + x$

$F(3) - F(1)$

$9 + 3 - \left(\frac{1}{3} + 1\right)$

$12 - \frac{4}{3}$

$\boxed{\frac{32}{3}}$



b) Approximate the area of R by dividing it into ten subintervals of equal width and summing the areas of rectangles.

width $= \frac{3-1}{10} = \frac{2}{10} = \frac{1}{5}$

height $= f(1 + \frac{i}{5})^2$

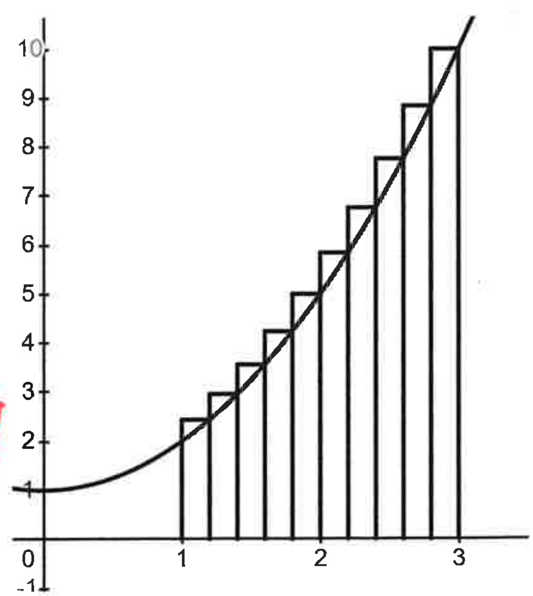
we start at 1 $\frac{5}{5} \rightarrow \frac{15}{5}$

$A = \frac{1}{5} \sum_{i=1}^{10} f(1 + \frac{i}{5})^2$

$A = \frac{1}{5}\left[(1+\frac{6}{5})^2 + (1+\frac{7}{5})^2 + (1+\frac{8}{5})^2 + \ldots + (1+\frac{15}{5})^2\right]$

$A = \frac{1}{5}\left[10 + \frac{1}{25}(6^2 + 7^2 + 8^2 + \ldots + 15^2)\right]$

$= 2 + \frac{1}{125}(1185) = \frac{1435}{125} = \boxed{11.48}$



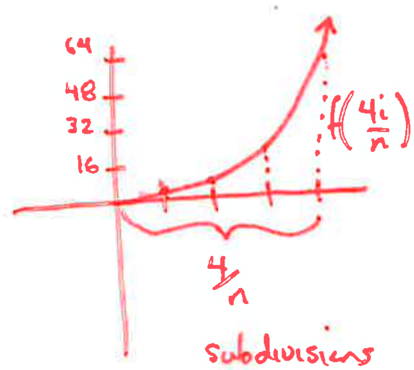
3. Use the methods of this section to calculate the area of the given region.

a) Under $y = x^3$ from 0 to 4

$\Delta x = 4n$ consider the function:

$x_i = \frac{4i}{n}$

$$A(\infty) = \sum_{i=1}^{n} \frac{4}{n}\cdot f\left(\frac{4i}{n}\right)$$



$$= \frac{4}{n}\sum_{i=1}^{n}\left(\frac{4i}{n}\right)^3 \to \frac{4}{n}\sum_{i=1} \frac{64i^3}{n^3}$$

$$\to \frac{256}{n^4}\sum_{i=1}^{n} i^3 \to \frac{256}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$$

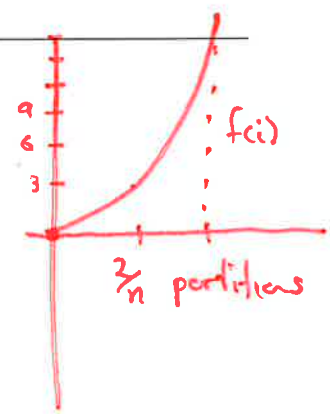
$$= \frac{64(n+1)^2}{n^2} = \frac{64(n^2+2n+1)}{n^2} \to 64\left(1+\frac{2}{n}+\frac{1}{n^2}\right)$$

$$A = 64\lim_{n\to}(1+0+0) = \boxed{64}$$

b) Under $y = x + 2x^3$ from 0 to 2

$\Delta x = 2/n$

$x_i = \frac{2i}{n}$



$$\sum_{i=1}^{n}\frac{2}{n} f\left(\frac{2i}{n}\right) = \sum_{i=1}^{n}\frac{2}{n}\left(\frac{2i}{n}+2\left(\frac{2i}{n}\right)^3\right)$$

$$= \sum_{i=1}^{n}\frac{2}{n}\left(\frac{2i}{n}+\frac{16i^3}{n^3}\right) \to \frac{2}{n}\left[\sum_{i=1}^{n}\frac{2i}{n}+\sum_{i=1}^{n}\frac{16i^3}{n^3}\right]$$

$$\to \frac{2}{n}\cdot\frac{2}{n}\sum_{i=1}^{n} i + \frac{2}{n}\cdot\frac{16}{n^3}\sum_{i=1}^{n} i^3$$

$$\to \frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{32}{n^4}\,\frac{(n^2(n+1)^2)}{4}$$

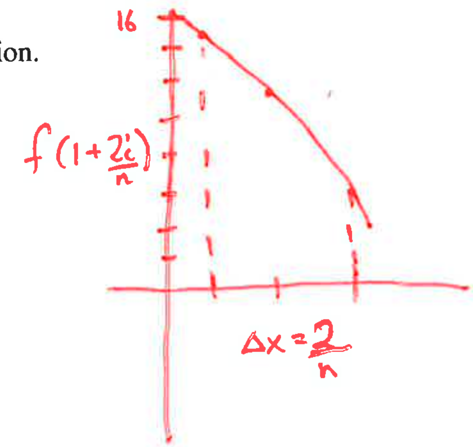
$$\frac{2n+2}{n} + \frac{8(n^2+2n+1)}{n^2}$$

$$\frac{2n^2+2n+8n^2+16n+8}{n^2}$$

$$\lim_{n\to\infty}\ 10+\frac{18}{n}+\frac{8}{n^2} = \boxed{10}$$

4. Use the methods of this section to calculate the area of the given region.

a) $y = -x^2 + 16$ from 1 to 3



$$\sum_{i=1}^{n} \frac{2}{n}\left(f\left(1+\frac{2i}{n}\right)\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n} -\left(1+\frac{2i}{n}\right)^2 - 16 \rightarrow -\frac{2}{n}\sum_{i=1}^{n} 1+\frac{4i}{n}+\frac{4i^2}{n^2}-16$$

$$\Rightarrow -\frac{2}{n}\left[\sum_{i=1}^{n} -15 + \sum_{i=1}^{n}\frac{4i}{n} + \sum_{i=1}^{n}\frac{4i^2}{n^2}\right] \rightarrow -\frac{2}{n}\left[-15n + \frac{4}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$\rightarrow -\frac{2}{n}\left[-15n + \frac{4n(n+1)}{2n} + \frac{2}{3}\frac{(n+1)(2n+1)}{n}\right] \rightarrow 30 - \frac{4(n+1)}{n} - \frac{4(n+1)(2n+1)}{3n^2}$$

$$30 - 4 - \frac{1}{n} - \frac{4(2n^2+3n+1)}{3n^2} = \lim_{n\to\infty} 30-4-0-\frac{8}{3}-\frac{12}{3n}-\frac{4}{3n^2} = \boxed{70/3}$$

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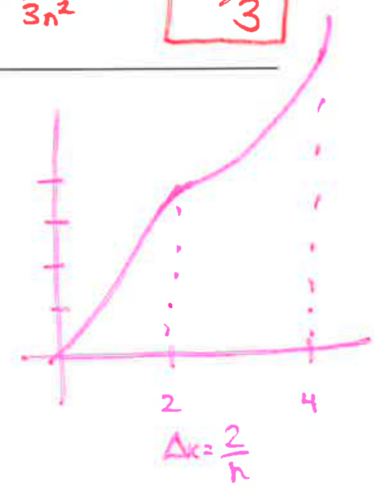
b) $y = \frac{1}{2}x^3$ from 2 to 4

$$f(x_i) = \left(2+\frac{2i}{n}\right)$$



$$\sum_{i=1}^{n}\frac{2}{n}\left(f\left(2+\frac{2i}{n}\right)\right)$$

$$\frac{2}{n}\sum_{i=1}^{n}\frac{1}{2}\left(2+\frac{2i}{n}\right)^3 \rightarrow \frac{1}{n}\sum_{i=1}^{n}\left(2+\frac{2i}{n}\right)^3$$

$$\rightarrow \frac{1}{n}\sum_{i=1}^{n} 8\left(1+\frac{i}{n}\right)^3 = \frac{8}{n}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3 \rightarrow \frac{8}{n}\sum_{i=1}^{n}\left(1+\frac{3i}{n}+\frac{3i^2}{n^2}+\frac{i^3}{n^3}\right)$$

$$\Rightarrow \frac{8}{n}\left[\sum_{i=1}^{n}1 + \frac{3}{n}\sum_{i=1}^{n} i + \frac{3}{n^2}\sum_{i=1}^{n} i^2 + \frac{1}{n^3}\sum_{i=1}^{n} i^3\right] \rightarrow \frac{8}{n}\left[n + \frac{3}{n}\frac{n(n+1)}{2} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3}\frac{n^2(n+1)^2}{4}\right]$$

$$\Rightarrow \frac{8}{n}\left[n + \frac{3(n+1)}{2} + \frac{3(n+1)(2n+1)}{6n} + \frac{(n+1)^2}{4n^2}\right] \rightarrow 8 + 12 + \frac{12}{n} + \frac{48}{6} + \frac{72}{n} + \frac{24}{n^2} + 2 + \frac{4}{n} + \frac{8}{n^2}$$

$$\lim_{n\to\infty} 8+12+\frac{12}{n}+8+\frac{72}{n}+\frac{24}{n^2}+2+\frac{4}{n}+\frac{8}{n^2} = \boxed{30}$$