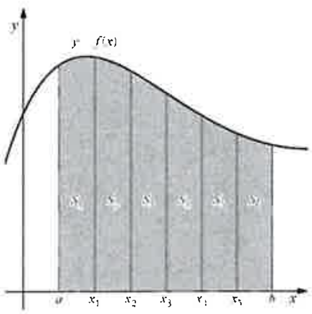
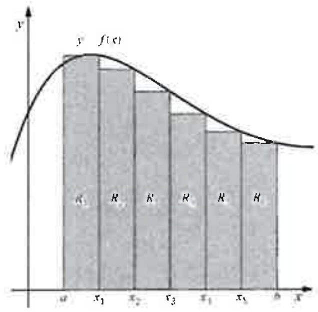
10.4 Areas as Limits

This section serves as a preview to Chapter 11 proper on integrals. We have simple formulas for calculating the area of regular polygons such as triangles or rectangles. But how do we calculate the area of a geometric figure such as a semicircle? If we imagine breaking up the shape into thin strips, we could approximate the area under a curve given by a function $y = f(x)$. The sum of the areas of the thin strips would then approximate the area under the curve.



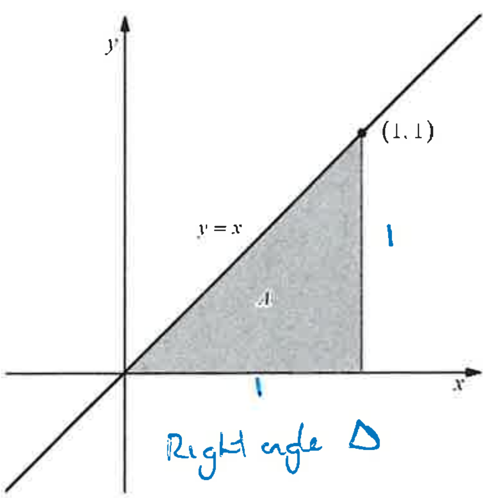


On the left, the region under the curve been divided into six strips, $S_1, S_2, \dots, S_6$. The problem remains of calculating the area underneath each strip as the top parts of each strip are still curved. The area can be estimated by instead separating the area into rectangular strips, $R_1, R_2, \dots, R_6$. In this case, the base of the triangle is the width of the strip, and the height of each rectangle is the value of the function at the *right-hand* endpoint of each strip. For example, rectangle R_2 has a width of $x_2 - x_1$ and height $f(x_2)$. The midpoint of the rectangle or the left-hand endpoint could have also been used to calculate the height of each strip (the result will be the same) but we will exclusively use the right-hand endpoint.

Ex. 1

(a) Calculate the area under $y = x$, from 0 to 1.
(b) Approximate the same area by subdividing the region into six strips of equal width and finding the sum of the areas of the rectangles determined by the right-hand endpoint of each interval.
(c) Repeat part (b) using twelve strips of equal width.

(a)

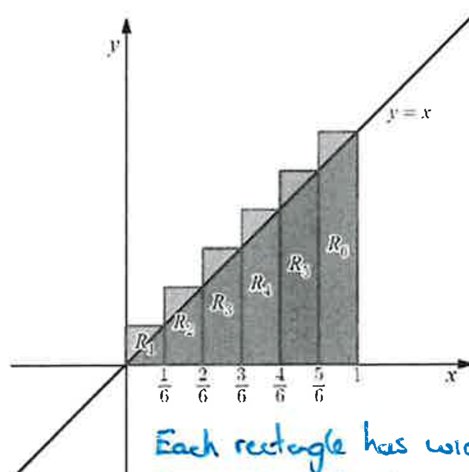


Right angle Δ

$A = \frac{1}{2}(1)(1)$

$= \frac{1}{2}$

(b)



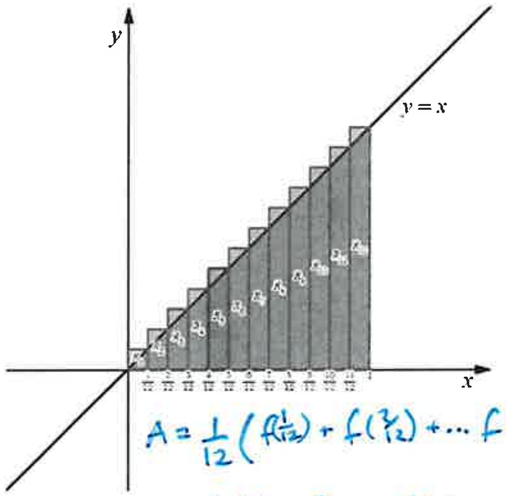
Each rectangle has width $\frac{1}{6}$ and height $f(x)$

$A = \frac{1}{6}(f(\frac{1}{6})) + \frac{1}{6}(f(\frac{2}{6})) + \frac{1}{6}(f(\frac{3}{6})) + \frac{1}{6}(f(\frac{4}{6})) + \dots$

$= \frac{1}{6}(\frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6})$

$= \frac{1}{6}(\frac{21}{6}) = \frac{21}{36} \doteq 0.583$

(c)



$$A = \frac{1}{12}\left(f\left(\tfrac{1}{12}\right) + f\left(\tfrac{2}{12}\right) + \dots f\left(\tfrac{12}{12}\right)\right)$$

$$= \frac{1}{12}\left(\frac{1}{12} + \frac{2}{12} + \dots \frac{12}{12}\right) = \frac{1}{12}\left(\frac{78}{12}\right) = \frac{78}{144} \doteq 0.542$$

(a) **ANSWER**
The required area is a right triangle with base 1 and height 1, so the area can be calculated using the formula for the area of a triangle.

$$A = \frac{1}{2}bh$$

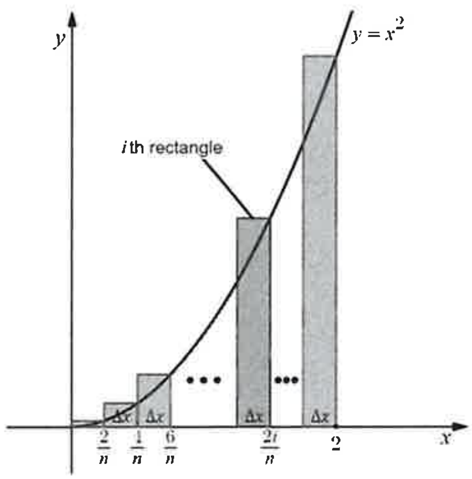
$$= \frac{1}{2}(1)(1)$$

$$= \frac{1}{2}$$

$$= 0.5$$

Ex. 2

Find the area under $y = x^2$ from $x = 0$ to $x = 2$.



$$\sum_{i=1}^{n} \frac{2}{n} f\left(\frac{2i}{n}\right)^2 = \sum_{i=1}^{n}\left(\frac{2}{n}\right)\left(\frac{4i^2}{n^2}\right)$$

$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

from 10 Review + Preview

$$= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{8(n+1)(2n+1)}{6n^2} = \frac{4(n+1)(2n+1)}{3n^2}$$

$$= \frac{4(2n^2+3n+1)}{3n^2}$$ as $n \to \infty$ (smaller and smaller partitions)

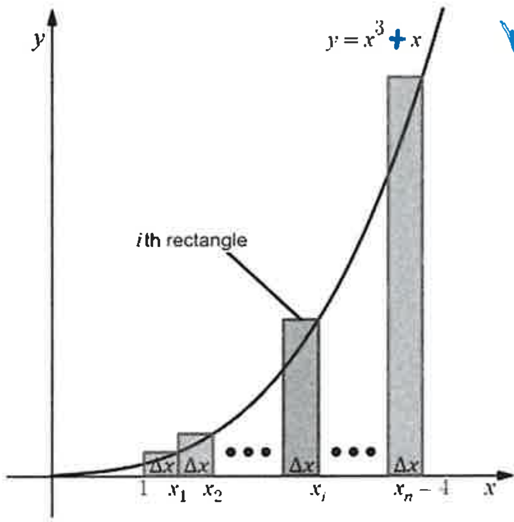
$$A = \frac{4}{3}\lim_{n\to\infty}\frac{2n^2+3n+1}{n^2}$$

$$= \frac{4}{3}\lim_{n\to\infty}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$= \frac{4}{3}(2+0+0)$$

$$= \boxed{8/3}$$

Ex. 3
Find the area under $y = x^3 + x$ from $x = 1$ to $x = 4$.



We subdivide our regions into n equal strips

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

$x_1 = 1 + \frac{3}{n}$

$x_2 = 1 + \frac{6}{n}$

$x_3 = 1 + \frac{9}{n}$

$\vdots$

$x_i = 1 + \frac{3i}{n}$

height is $f\left(1 + \frac{3i}{n}\right)$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n} f\left(1 + \frac{3i}{n}\right)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(1 + \frac{3i}{n}\right)^3 + 1 + \frac{3i}{n}\right]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n}\left[1 + \frac{9i}{n} + \frac{27i^2}{n^2} + \frac{27i^3}{n^3} + 1 + \frac{3i}{n}\right]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left[\frac{3}{n} + \frac{27i}{n^2} + \frac{81i^2}{n^3} + \frac{81i^3}{n^4} + \frac{3}{n} + \frac{9i}{n^2}\right]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left[\frac{81i^3}{n^4} + \frac{81i^2}{n^3} + \frac{36i}{n^2} + \frac{6}{n}\right]$$

$$= \lim_{n \to \infty} \left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 + \frac{81}{n^3}\sum_{i=1}^{n} i^2 + \frac{36}{n}\sum_{i=1}^{n} i + \frac{6}{n}\sum_{i=1}^{n} 1\right]$$

$$= \lim_{n \to \infty} \left[\frac{81n^2(n+1)^2}{4n^4} + \frac{81n(n+1)(2n+1)}{6n^3} + \frac{36n(n+1)}{2n^2} + \frac{6n}{n}\right]$$

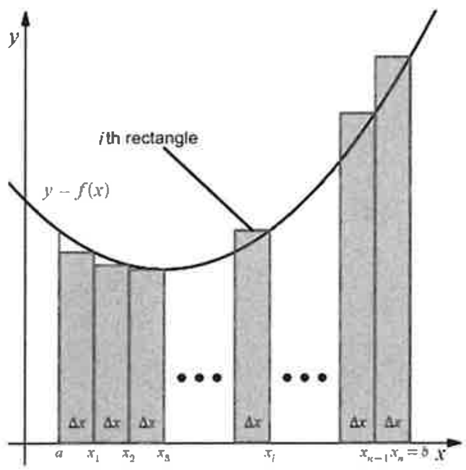
$$= \lim_{n \to \infty} \left[\frac{81(n^2+2n+1)}{4n^2} + \frac{81(2n^2+3n+1)}{6n^2} + \frac{36(n+1)}{2n} + 6\right] \Rightarrow \frac{81}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) + \frac{81}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 18\left(1 + \frac{1}{n}\right) + 6$$

$$\frac{81}{4} + \frac{81}{3} + 18 + 6$$

$$= \boxed{\frac{285}{4}}$$

A formula to calculate the area under any function $y = f(x)$ from a to b for f continuous and positive. To start the region is subdivided into n strips of equal width Δx. Where Δx is given by the following equation.

$$\Delta x = \frac{b-a}{n}$$



From the diagram we see that the right-hand endpoints of the intervals are:

$$x_1 = a + \Delta x$$
$$x_2 = a + 2\Delta x$$
$$x_3 = a + 3\Delta x$$
$$\vdots$$

The right-hand endpoint of the ith interval is:

$$x_i = a + i\Delta x$$

The height of the ith rectangle is $f(x_i)$, so its area can be calculated using the following formula.

$$f(x_i)\Delta x = \text{height} \times \text{width}$$

To find the required area, take the limit of the sums as n approaches infinity of the areas of the rectangles.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

Homework Assignment

- Practice Problems: #1-4